YOUR NAME PLEASE:

* SOLUTIONS *

Computer Science 202
Final Exam
2-5 pm, 20 December 2015

Closed book and closed notes. No electronic devices. Show all written work on the test itself, using the backs of pages as necessary.

For problems that do not ask you to justify the answer, an answer alone is sufficient. However, if the answer is wrong and no derivation or supporting reasoning is given, there will be no partial credit.

This is a two-and-a-half hour exam, with an additional half an hour “for improving what has already been written.” It will be collected at 5 pm.

GOOD LUCK!

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1. (10 points) Translate each of the following statements into a predicate logic formula. You may use functions \( \cdot \) and \( + \), and predicates \( =, <, \leq, >, \geq \), as well as \( p(n) \) for "\( n \) is prime number" and \( q(n) \) for "\( n \) is a power of 2". Assume that the domain is all integers greater than or equal to 0.

(a) There exists a number that is less than or equal to every number.
\[
\exists x \, \forall y \, (x \leq y)
\]

(b) For every number there exists a prime number greater than it.
\[
\forall x \, \exists y \, (p(y) \land (x < y))
\]

(c) The product of two numbers is not necessarily greater than their sum.
\[
\exists x \, \exists y \, (x \cdot y \leq x + y)
\]

(d) The product of two prime numbers is never a prime number.
\[
\forall x \, \forall y \, (p(x) \land p(y) \rightarrow \neg p(x \cdot y))
\]

(e) There exists exactly one number that is both a power of two and a prime number.
\[
\exists x \, (p(x) \land q(x) \land \forall y \, (p(y) \land q(y) \rightarrow x = y))
\]
\[
\lor \exists x \, \forall y \, (p(y) \land q(y) \leftrightarrow x = y)
\]
2. (10 points) Give succinct definitions of the following terms. Assume $A$, $B$ and $C$ are sets, and $f : A \to B$ and $g : B \to C$ are functions.

(a) The function $f$ is injective.

For all $a_1, a_2 \in A$, if $a_1 \neq a_2$
then $f(a_1) \neq f(a_2)$.

(b) The set $A$ is a subset of the set $B$.

For all $x$, $x \in A$ implies $x \in B$.

(c) The composition $(g \circ f)$ of the functions $f$ and $g$.

$(g \circ f)$ is the function with domain $A$
and co-domain $C$ such that
for all $a \in A$, $(g \circ f)(a) = g(f(a))$.

(d) The sets $A$, $B$, and $C$ are pairwise disjoint.

$A \cap B = B \cap C = A \cap C = \emptyset$

(e) $|A| = |B|$. (Your definition must work for infinite sets.)

There exists a bijection $f : A \to B$. 
3. (10 points) Prove by simple mathematical induction that $2^n > 3n$ for all integers $n \geq 4$. Please identify the predicate $P(n)$, the base case(s), and the induction hypothesis.

The predicate $P(n)$ is $2^n > 3n$.

**Base case:** $n = 4$.

$2^4 = 16 > 12 = 3 \cdot 4$, so $P(4)$ is true.

**Inductive hypothesis:** For some $k \geq 4$,

$P(k)$ is true, that is, $2^k > 3k$.

Then

$2^{k+1} = 2 \cdot 2^k$

$> 2 \cdot (3k)$ by the inductive hypothesis

$> 3k + 3k$

$> 3k + 3$ because $k \geq 4$,

$> 3(k+1)$

Thus $P(k) \rightarrow P(k+1)$, so by induction we conclude

$2^n > 3n$ for all $n \geq 4$. 

4. (a) (5 points) Find integers \( s \) and \( t \) such that \( 1 = 13s + 44t \), using the Euclidean algorithm for the \( \text{gcd} \).

\[
\gcd(13, 44) \\
\begin{array}{c}
13 \div 44 \\
\underline{3 \times 44} \\
39 \\
\underline{5} \\
9 \\
\underline{3} \\
6 \\
\underline{2}
\end{array}
\]

Thus:
\[
\begin{align*}
1 & = 2 \cdot 13 - 5 \cdot 44 \\
& = 17 \cdot 13 - 5 \cdot 44
\end{align*}
\]

\[
\gcd(5, 13) \\
\begin{array}{c}
5 \div 13 \\
\underline{2 \times 13} \\
10 \\
\underline{3} \\
7 \\
\underline{2}
\end{array}
\]

\[
\begin{align*}
1 & = 2 \cdot (13 - 2 \cdot 5) - 3 \\
& = 2 \cdot 13 - 5 \cdot 5
\end{align*}
\]

\[
\begin{align*}
\gcd(3, 5) & = 3 - (5 - 3) \\
& = 2 - 5
\end{align*}
\]

\[
\gcd(2, 3) & = 1 - 2
\]

(b) (5 points) Find the smallest positive integer solution \( x \) of the following equation, justifying your steps.

\[
13x + 5 \equiv 9 \pmod{44}.
\]

Subtract 5 from both sides:

\[
13x \equiv 4 \pmod{44}
\]

Multiply both sides by 17 (because \( 17 \cdot 13 \equiv 1 \pmod{44} \)):

\[
x \equiv 68 \pmod{44}
\]

\[
x \equiv 24 \pmod{44}
\]

Thus \( x = 24 \) is the solution.
5. Let $F$ be the set of all functions $f(n)$ with domain and co-domain $\mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers (positive, negative and zero).

(a) (4 points) Let $f(n), g(n) \in F$. Give the definition of: $f(n)$ is in $O(g(n))$.

$f(n)$ is in $O(g(n))$ iff there exist real numbers $c > 0$ and $N_0$ such that

$$ |f(n)| \leq c |g(n)| \text{ for all } n \geq N_0.$$

(b) (4 points) Define the binary relation $R$ on $F$ by $(f(n), g(n)) \in R$ if and only if $f(n)$ is in $O(g(n))$. Prove that $R$ is transitive.

Assume $(f(n), g(n)) \in R$ and $(g(n), h(n)) \in R$.

Then there exist real numbers $c > 0$ and $N_0$ such that $|f(n)| \leq c |g(n)|$ for all $n \geq N_0$ and also real numbers $d > 0$ and $N_1$ such that $|g(n)| \leq d |h(n)|$ for all $n \geq N_1$.

Thus, if $n \geq \max\{N_0, N_1\}$ then

$$ |f(n)| \leq c |g(n)| \leq c \cdot d |h(n)|,$$

and $c \cdot d > 0$, so $f(n)$ is in $O(h(n))$ and

$(f(n), h(n)) \in R$. Thus $R$ is transitive.

(c) (2 points) Is the relation $R$ in part (b) a partial order? Why or why not?

No, $R$ is not a partial order.

While $R$ is clearly reflexive, and is transitive by (b), it is not antisymmetric.

For example, if $f(n) = n$ and $g(n) = 2n$,

then $(f(n), g(n)) \in R$ and $(g(n), f(n)) \in R$,

but $f(n) \neq g(n)$. 
6. (a) (4 points) Let $G = (V, E)$ be a simple undirected graph. State the Handshaking Lemma, giving the relationship between the sum of the degrees of the vertices of $G$ and the number of edges of $G$.

\[ \sum_{v \in V} \deg(v) = 2|E| \]

(or: the sum of the degrees of the vertices of $G$ is equal to twice the number of edges of $G$.)

(b) (6 points) Suppose that $T = (V, E)$ is a tree and $F \subseteq E$ is a set of $k$ edges of $T$. We define the graph $G = (V, E \setminus F)$. Let $V_1, V_2, \ldots, V_\ell$ be the sets of vertices in the connected components of $G$. For all $i = 1, 2, \ldots, \ell$, let $T_i$ be the subgraph of $G$ induced by the vertices $V_i$. Prove that $T_i$ is a tree for all $i = 1, 2, \ldots, \ell$, and prove that $\ell = k + 1$.

Each $T_i$ is connected, because $V_i$ is a connected component of $G$. Each $T_i$ is acyclic because a cycle in $T_i$ would also be a cycle in $T$, which is acyclic because it is a tree. Thus, each $T_i$ is connected and acyclic, and therefore a tree.

Thus, the number of edges in $T_i$ is $|V_i| - 1$ and the total number of edges in all the $T_i$'s is

\[ \sum_{i=1}^{\ell} (|V_i| - 1) = |V| - \ell. \]

But this must equal the total number of edges in $G$, which is $|E| - k$, or $|V| - 1 - k$, because $T$ is a tree.

Thus $|V| - \ell = |V| - (k + 1)$, so $\ell = k + 1$. 
7. (a) (3 points) State the finite version of the Binomial Theorem, which gives an expression for the value of \((x + y)^n\).

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

(b) Consider the following identity:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}
\]

i. (2 points) Verify this identity for \(n = 4\), showing your work.

\[
\begin{align*}
\binom{4}{0} \cdot \binom{4}{4} &+ \binom{4}{1} \binom{4}{3} + \binom{4}{2} \binom{4}{2} + \binom{4}{3} \binom{4}{1} + \binom{4}{4} \binom{4}{0} \\
= 1 \cdot 1 &+ 4 \cdot 4 + 6 \cdot 6 + 4 \cdot 4 + 1 \cdot 1 \\
= 70
\end{align*}
\]

ii. (5 points) Give a combinatorial proof of the identity for all integers \(n \geq 1\). Let \(S = \{1, 2, \ldots, 2n\}\). \(S\) contains \(n\) even numbers and \(n\) odd numbers.

\(\binom{2n}{n}\) is the number of sets \(T \subseteq S\) of cardinality \(n\).

We can group the sets \(T\) by \(k\) = the number of even numbers \(T\) contains; then \(T\) contains \((n-k)\) odd numbers.

By the product principle, there are \(\binom{n}{k} \binom{n}{n-k}\) sets \(T\) with \(k\) even numbers.

Because these groups are pairwise disjoint, the total number of sets \(T \subseteq S\) of cardinality \(n\) is

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}
\]
8. Suppose there are two boxes: A containing 4 balls (2 red, 1 yellow and 1 green), and B containing 5 balls (1 red, 3 yellow and 1 green). I select one of the two boxes uniformly at random (noting which one) and then select from that box one of the balls it contains, again uniformly at random (noting its color). For example, if I choose box A and choose a red ball from that box, then the outcome is \((A, R)\).

(a) (3 points) Give the 6 possible outcomes of this experiment and their probabilities.

\[
\begin{align*}
Pr( A, R ) &= \frac{1}{2} \cdot \frac{2}{4} = \frac{1}{4} \\
Pr( A, Y ) &= \frac{1}{2} - \frac{1}{4} = \frac{1}{8} \\
Pr( A, G ) &= \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \\
Pr( B, R ) &= \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10} \\
Pr( B, Y ) &= \frac{1}{2} - \frac{3}{5} = \frac{3}{10} \\
Pr( B, G ) &= \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}
\end{align*}
\]

(b) (4 points) If \(A\) is the event that box A is chosen and \(R\) is the event that a red ball is chosen, calculate the probabilities \(Pr[A], Pr[R], Pr[A \cap R]\). Are \(A\) and \(R\) independent events? Justify your answer.

\[
\begin{align*}
Pr[A] &= \frac{1}{2} \\
Pr[R] &= Pr\left[ \{A, R\} \cup \{B, R\} \right] = \frac{1}{20} \\
Pr[A \cap R] &= \frac{1}{4}
\end{align*}
\]

Then \(Pr[A] \cdot Pr[R] = \frac{7}{40}\) while \(Pr[A \cap R] = \frac{1}{4} \neq \frac{7}{40}\)

So the events \(A\) and \(R\) are not independent.

(c) (3 points) Suppose someone argues that since there is just one green ball in each box, knowing that a green ball was selected should not change our estimate that boxes \(A\) and \(B\) were chosen with equal probability. Is this correct? Justify your answer.

This is incorrect. We show \(Pr[A | G] \neq \frac{1}{2}\) as follows:

\[
Pr[A | G] = \frac{Pr[A \cap G]}{Pr[G]} = \frac{\frac{1}{8}}{\frac{9}{40}} = \frac{5}{9} > \frac{1}{2}
\]

because \(Pr[G] = Pr\left[ \{A, G\}, \{B, G\} \right] = \frac{9}{40}\)
9. (a) (3 points) Let \( S = \{ \overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k \} \) be a set of vectors from \( \mathbb{R}^n \). Give definitions of the concepts: (i) \( \overline{u} \) is a linear combination of vectors from \( S \), (ii) \( S \) is linearly dependent, and (iii) the span of \( S \).

(i) \( \overline{u} = c_1 \overline{v}_1 + c_2 \overline{v}_2 + \cdots + c_k \overline{v}_k \) for some \( c_1, c_2, \ldots, c_k \in \mathbb{R} \).

(ii) There exist \( c_1, c_2, \ldots, c_k \in \mathbb{R} \) not all 0 such that \( c_1 \overline{v}_1 + c_2 \overline{v}_2 + \cdots + c_k \overline{v}_k = \overline{0} \).

(iii) The span of \( S \) is the set of all linear combinations of vectors from \( S \).

(b) (7 points) Let \( A \) be the matrix given by:

\[
A = \begin{pmatrix}
1 & -1 & 0 \\
2 & 0 & 4 \\
3 & -1 & 0
\end{pmatrix}
\]

Compute the inverse of \( A \) using Gauss-Jordan elimination, indicating the (one or two) row operations performed in each step. Indicate what \( A^{-1} \) is.

\[
R_1 = R_2 - 2R_1
\]

\[
R_3 = R_3 - 3R_1
\]

\[
\rightarrow
\]

\[
R_2 = \frac{R_2}{2}
\]

\[
R_1 = R_1 + R_2
\]

\[
R_3 = 2R_2
\]

\[
\rightarrow
\]

\[
R_3 = \frac{R_3}{4}
\]

\[
A^{-1} = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 2 \\
0 & 0 & -4
\end{pmatrix}
\]

10.
10. (a) (4 points) Draw the directed graph $G$ whose vertices are $\{1, 2, 3, 4\}$ and whose adjacency matrix $A$ is:

$$A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}$$

and compute the matrix product $A \cdot A$.

(b) (6 points) Let $n > 0$ be an integer and let $G$ be an arbitrary directed graph with vertices $V = \{1, 2, \ldots, n\}$ and adjacency matrix $A$. We inductively define $A^1 = A$ and $A^{k+1} = A^k \cdot A$. Prove that for any integer $\ell > 0$, the value of $(A^\ell)_{ij}$ is the number of directed paths of length $\ell$ in $G$ from vertex $i$ to vertex $j$.

Proof by simple mathematical induction.

$P(1) =$ For all vertices $i, j \in V$,

$(A^1)_{ij}$ is the number of directed paths of length $1$ in $G$ from $i$ to $j$.

Basis $P(1)$. If $(A^1)_{ij} = 1$ then there is an edge from $i$ to $j$ and exactly one path of length $1$ from $i$ to $j$. If $(A^1)_{ij} = 0$ then there is no edge from $i$ to $j$ and no paths of length $1$ from $i$ to $j$. Thus $P(1)$ is true.

Inductive hypothesis: $P(\ell)$ holds for some $\ell \geq 1$.

Then $(A^{\ell+1})_{ij} = \sum_{k=1}^{n} (A^\ell)_{ik} \cdot A_{kj}$. The number of paths of length $\ell+1$ from $i$ to $j$ is the sum over all vertices $k$ of the number of paths of length $\ell$ from $i$ to $k$, such that there is an edge from $k$ to $j$, which is $(A^\ell)_{ik} \cdot A_{kj}$ by the induction hypothesis. Thus $P(\ell) \rightarrow P(\ell+1)$ and we conclude $P(\ell)$ is true for all $\ell \geq 1$. 