YOUR NAME PLEASE:

Computer Science 202
Final Exam
2-5 pm, 20 December 2015

Closed book and closed notes. No electronic devices. Show all written work on the test itself, using the backs of pages as necessary.

For problems that do not ask you to justify the answer, an answer alone is sufficient. However, if the answer is wrong and no derivation or supporting reasoning is given, there will be no partial credit.

This is a two-and-a-half hour exam, with an additional half an hour “for improving what has already been written.” It will be collected at 5 pm.

GOOD LUCK!

<table>
<thead>
<tr>
<th>problem</th>
<th>possible</th>
<th>score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points) Translate each of the following statements into a predicate logic formula. You may use functions · and +, and predicates =, <, ≤, >, ≥, as well as p(n) for “n is prime number” and q(n) for “n is a power of 2”. Assume that the domain is all integers greater than or equal to 0.

(a) There exists a number that is less than or equal to every number.

(b) For every number there exists a prime number greater than it.

(c) The product of two numbers is not necessarily greater than their sum.

(d) The product of two prime numbers is never a prime number.

(e) There exists exactly one number that is both a power of two and a prime number.
2. (10 points) Give succinct definitions of the following terms. Assume $A$, $B$ and $C$ are sets, and $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.

(a) The function $f$ is injective.

(b) The set $A$ is a subset of the set $B$.

(c) The composition $(g \circ f)$ of the functions $f$ and $g$.

(d) The sets $A$, $B$, and $C$ are pairwise disjoint.

(e) $|A| = |B|$. (Your definition must work for infinite sets.)
3. (10 points) Prove by simple mathematical induction that $2^n > 3n$ for all integers $n \geq 4$. Please identify the predicate $P(n)$, the base case(s), and the induction hypothesis.
4. (a) (5 points) Find integers $s$ and $t$ such that $1 = 13s + 44t$, using the Euclidean algorithm for the gcd.

(b) (5 points) Find the smallest positive integer solution $x$ of the following equation, justifying your steps.

$$13x + 5 \equiv 9 \pmod{44}.$$
5. Let $F$ be the set of all functions $f(n)$ with domain and co-domain $\mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers (positive, negative and zero).

(a) (4 points) Let $f(n), g(n) \in F$. Give the definition of: $f(n)$ is in $O(g(n))$.

(b) (4 points) Define the binary relation $R$ on $F$ by $(f(n), g(n)) \in R$ if and only if $f(n)$ is in $O(g(n))$. Prove that $R$ is transitive.

(c) (2 points) Is the relation $R$ in part (b) a partial order? Why or why not?
6. (a) (4 points) Let $G = (V, E)$ be a simple undirected graph. State the Handshaking Lemma, giving the relationship between the sum of the degrees of the vertices of $G$ and the number of edges of $G$.

(b) (6 points) Suppose that $T = (V, E)$ is a tree and $F \subseteq E$ is a set of $k$ edges of $T$. We define the graph $G = (V, E \setminus F)$. Let $V_1, V_2, \ldots V_\ell$ be the sets of vertices in the connected components of $G$. For all $i = 1, 2, \ldots, \ell$, let $T_i$ be the subgraph of $G$ induced by the vertices $V_i$. Prove that $T_i$ is a tree for all $i = 1, 2, \ldots, \ell$, and prove that $\ell = k + 1$. 
7. (a) (3 points) State the finite version of the Binomial Theorem, which gives an expression for the value of \((x + y)^n\).

(b) Consider the following identity:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \binom{2n}{n}.
\]

i. (2 points) Verify this identity for \(n = 4\), showing your work.

ii. (5 points) Give a combinatorial proof of the identity for all integers \(n \geq 1\).
8. Suppose there are two boxes: $A$ containing 4 balls (2 red, 1 yellow and 1 green), and $B$ containing 5 balls (1 red, 3 yellow and 1 green). I select one of the two boxes uniformly at random (noting which one) and then select from that box one of the balls it contains, again uniformly at random (noting its color). For example, if I choose box $A$ and choose a red ball from that box, then the outcome is $(A, R)$.

(a) (3 points) Give the 6 possible outcomes of this experiment and their probabilities.

(b) (4 points) If $A$ is the event that box $A$ is chosen and $R$ is the event that a red ball is chosen, calculate the probabilities $\Pr[A]$, $\Pr[R]$, $\Pr[A \cap R]$. Are $A$ and $R$ independent events? Justify your answer.

(c) (3 points) Suppose someone argues that since there is just one green ball in each box, knowing that a green ball was selected should not change our estimate that boxes $A$ and $B$ were chosen with equal probability. Is this correct? Justify your answer.
9. (a) (3 points) Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of vectors from $\mathbb{R}^n$. Give definitions of the concepts: (i) $\vec{u}$ is a linear combination of vectors from $S$, (ii) $S$ is linearly dependent, and (iii) the span of $S$.

(b) (7 points) Let $A$ be the matrix given by:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 4 \\ 3 & -1 & 0 \end{pmatrix}$$

Compute the inverse of $A$ using Gauss-Jordan elimination, indicating the (one or two) row operations performed in each step. Indicate what $A^{-1}$ is.
10. (a) (4 points) Draw the directed graph \( G \) whose vertices are \( \{1, 2, 3, 4\} \) and whose adjacency matrix \( A \) is:

\[
A = \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\end{pmatrix}
\]

and compute the matrix product \( A \cdot A \).

(b) (6 points) Let \( n > 0 \) be an integer and let \( G \) be an arbitrary directed graph with vertices \( V = \{1, 2, \ldots, n\} \) and adjacency matrix \( A \). We inductively define \( A^1 = A \) and \( A^{k+1} = A^k \cdot A \). Prove that for any integer \( \ell > 0 \), the value of \((A^\ell)_{ij}\) is the number of directed paths of length \( \ell \) in \( G \) from vertex \( i \) to vertex \( j \).