HW 10 SOLUTIONS
(as of 12/16/15)

(a)

The event $X_2 = 1$ contains the $3! = 6$ permutations in which 2 is in position 2, that is, all possible ways of choosing $d_1, d_2, d_3$ where $d_1 d_2 d_3$ is a permutation of 134.

Thus, $Pr[X_2 = 1] = \frac{6}{24} = \frac{1}{4}$ (Similarly,

$E[X_2] = 1 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = \frac{1}{4}$ for $i = 1, 3, 4$)

(b)
The event $X_1 = 1 \land X_3 = 1$ contains the two permutations 1234 and 1432, in which 1 is in position 1 and 3 is in position 3.

Thus,

$Pr[X_1 = 1 \land X_3 = 1] = \frac{2}{24} = \frac{1}{12}$

(c) By the definition of conditional probability, since $Pr[X_3 = 1] = \frac{1}{4}$ (as in 1(a)), we have

$Pr[X_1 = 1 | X_3 = 1] = \frac{Pr[X_1 = 1 \land X_3 = 1]}{Pr[X_3 = 1]} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$

(d) No, the events $X_1 = 1$ and $X_3 = 1$ are not independent, because

$Pr[X_1 = 1 \land X_3 = 1] = \frac{1}{12} \neq \frac{1}{16} = Pr[X_1 = 1] \cdot Pr[X_3 = 1].$

(e) $E[X_1 + X_2 + X_3 + X_4] = E[X_1] + E[X_2] + E[X_3] + E[X_4]$

(by linearity of expectation)

$= 4 \cdot \frac{1}{4}$

$= 1$
2. X is binomially distributed, \( p = 0.6 \) and \( n = 4 \)

(a) Table for X

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \Pr[X = k] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \binom{4}{0} (0.6)^0 (0.4)^4 = 0.0256 )</td>
</tr>
<tr>
<td>1</td>
<td>( \binom{4}{1} (0.6)^1 (0.4)^3 = 0.1536 )</td>
</tr>
<tr>
<td>2</td>
<td>( \binom{4}{2} (0.6)^2 (0.4)^2 = 0.3456 )</td>
</tr>
<tr>
<td>3</td>
<td>( \binom{4}{3} (0.6)^3 (0.4)^1 = 0.3456 )</td>
</tr>
<tr>
<td>4</td>
<td>( \binom{4}{4} (0.6)^4 (0.4)^0 = 0.0256 )</td>
</tr>
</tbody>
</table>

(b) \( \Pr[X = 1] > \Pr[X = 4] \)

(c) The expected value of a binomially distributed random variable with parameters \( n \) and \( p \) is \( np \).

Thus

\[ E[X] = 4 \cdot 0.6 = 2.4 \]

The variance of a binomially distributed random variable with parameters \( n \) and \( p \) is \( np(1-p) \).

Thus

\[ \text{Var}[X] = 4 \cdot 0.6 \cdot 0.4 = 0.96 \]
3. Coin C₁ has a probability of heads of 0.5
   Coin C₂ has a probability of heads of 0.2

   I choose one of the two coins uniformly at random, flip it 3 times and observe 3 heads.
   What is the posterior probability of choosing coin C₁, given the observation that 3 flips
   produced 3 heads?

   Let Cᵢ be the event that I chose coin Cᵢ, for i = 1, 2.
   Then Pr(C₁) = 0.5, Pr(C₂) = 0.5
   Let H₃ be the event that 3 flips produce 3 heads.

   Then Pr(H₃ | C₁) = (0.7)³ = 0.343
   Pr(H₃ | C₂) = (0.2)³ = 0.008

   We use Bayes' Theorem to find Pr(C₁ | H₃).

   \[
   Pr(C₁ | H₃) = \frac{Pr(H₃ | C₁) Pr(C₁)}{Pr(H₃)} = \frac{(0.343) \cdot (0.5)}{Pr(H₃ | C₁) Pr(C₁) + Pr(H₃ | C₂) Pr(C₂)}
   \]

   \[
   = \frac{(0.343) \cdot (0.5)}{(0.343) \cdot (0.5) + (0.008) \cdot (0.5)}
   \]

   \[
   = 0.917 \quad \text{(to 3 digits)}
   \]
4. Note that

\[
A^1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.
\]

\[
A^2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.
\]

\[
A^3 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}.
\]

Conjecture that for all integers \( m \geq 1 \),

\[
A^m = \begin{pmatrix} m+1 & -m \\ m & -(m-1) \end{pmatrix}.
\]

This is \( P(m) \).

**Base case:** \( A^1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1 & -1 \\ 1 & -(1-1) \end{pmatrix} \).

Assume for some \( m \geq 1 \), \( P(m) \) holds. (This is IH.)

By definition,

\[
A^{m+1} = A^m \cdot A
\]

\[
= \begin{pmatrix} m+1 & -m \\ m & -(m-1) \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}
\]

by IH

\[
= \begin{pmatrix} 2m+2-m & -(m+1) \\ 2m-(m-1) & -m \end{pmatrix}
\]

by definition of matrix product

\[
= \begin{pmatrix} m+2 & -(m+1) \\ m+1 & -m \end{pmatrix}
\]

So \( P(m) \Rightarrow P(m+1) \). By induction, \( P(m) \) holds for all integers \( m \geq 1 \).
5. Find a basis for the vector space spanned by the vectors

\[ \{ (6, 9, 3), (9, 14, 5), (-3, -4, -1) \} \]

There are several ways to solve this. We perform row operations on a matrix with the given vectors as rows:

\[
\begin{pmatrix}
6 & 9 & 3 \\
9 & 14 & 5 \\
-3 & -4 & -1
\end{pmatrix}
\]

Add 2 times the 3rd row to the 1st row:

\[
\begin{pmatrix}
0 & 1 & 1 \\
9 & 14 & 5 \\
-3 & -4 & -1
\end{pmatrix}
\]

Add 3 times the 3rd row to the 2nd row:

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 2 & 2 \\
-3 & -4 & -1
\end{pmatrix}
\]

Observe that the second row is twice the first. One basis is the two vectors:

\[ \{ (0, 1, 1), (3, 4, 1) \} \]

They are independent because neither is a multiple of the other. They span the space of the original vectors because row operations preserve the space spanned by the rows.

Note that:

\[
\begin{align*}
(6, 9, 3) &= 2 \cdot (3, 4, 1) + (0, 1, 1) \\
(9, 14, 5) &= 3 \cdot (3, 4, 1) + 2 \cdot (0, 1, 1) \\
(-3, -4, -1) &= -1 \cdot (3, 4, 1) + 0 \cdot (0, 1, 1)
\end{align*}
\]
6.

\[
\begin{pmatrix}
3 & 4 & 1 \\
6 & 9 & 3 \\
3 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

subtract twice the first row from the second row

\[
\begin{pmatrix}
3 & 4 & 1 \\
0 & 1 & 1 \\
3 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

subtract the first row from the third row

\[
\begin{pmatrix}
3 & 4 & 1 \\
0 & 1 & 1 \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

add -4 times the second row to the first row

\[
\begin{pmatrix}
3 & 0 & -3 \\
0 & 1 & 1 \\
0 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
9 & -4 & 0 \\
-2 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

add 3 times the second row to the third row

\[
\begin{pmatrix}
2 & 0 & -3 \\
0 & 1 & 1 \\
0 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
9 & -4 & 0 \\
-2 & 1 & 0 \\
-7 & 3 & 1
\end{pmatrix}
\]

divide the first row by 3

\[
\begin{pmatrix}
10 & -1 \\
0 & 1 \\
0 & 0 & 4
\end{pmatrix}
\begin{pmatrix}
-\frac{4}{3} & 0 \\
-2 & 1 & 0 \\
-7 & 3 & 1
\end{pmatrix}
\]
6 continued)

divide the last row by 4

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
-3 & -\frac{1}{2} & 0 \\
2 & 1 & 0 \\
-\frac{1}{4} & \frac{3}{4} & \frac{1}{4}
\end{pmatrix}
\]

add third row to first row

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
\frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{3}{4} & \frac{1}{4}
\end{pmatrix}
\]

subtract third row from second row

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
\frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{3}{4} & \frac{1}{4}
\end{pmatrix}
\]

Check result

\[
\begin{pmatrix}
3 & 4 & 1 \\
6 & 9 & 3 \\
3 & 1 & 2
\end{pmatrix}
\times
\begin{pmatrix}
\frac{5}{4} & -\frac{1}{2} & \frac{1}{4} \\
-\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & \frac{3}{4} & \frac{1}{4}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]