Please submit your homework in **two separate parts**: problems 1-3 in the first part, and 4-6 in the second part. Make sure your name is on both parts. In the first part, please include the names of anyone (including course staff) you consulted with in connection with this assignment, as well as listing any resources (including course materials) you consulted. Please see Chapter 12 for the definitions of expectation and variance.

1. (20 points)
   Consider the outcome space $\Omega$ of all permutations of the numbers 1, 2, 3, and 4, where each permutation has probability $1/24$. Define the random variable $X_i$ to be 1 if the number $i$ is in the $i^{th}$ place, and 0 otherwise. For example, $X_1(3214) = 0$, $X_2(3214) = 1$, $X_3(3214) = 0$ and $X_4(3214) = 1$. Please give justifications for your answers to the following.

   (a) What is $\Pr[X_2 = 1]$? What is $E[X_2]$?
   (b) What is $\Pr[X_1 = 1 \land X_3 = 1]$?
   (c) What is $\Pr[X_1 = 1 | X_3 = 1]$?
   (d) Is the event $X_1 = 1$ independent of the event $X_3 = 1$?
   (e) What is $E[X_1 + X_2 + X_3 + X_4]$?

2. (15 points) Let $X$ be a binomially distributed random variable with $p = 0.6$ and $n = 4$.

   (a) Give a table of the values (as decimal fractions) of the probability distribution for $X$.
   (b) Which is larger, $\Pr[X = 1]$ or $\Pr[X = 4]$?
   (c) What are the expectation and variance of $X$, and why?

3. (15 points) Suppose there are two identical looking coins, $C_1$ with a probability of heads of 0.7 and $C_2$ with a probability of heads of 0.2. I cannot tell the coins apart, but I select one of the two coins at random, flip it three times and observe three heads. Initially the probability was 0.5 that the coin I selected was $C_1$. Apply Bayes’ Theorem to calculate the probability that the coin I selected was $C_1$, given that I flipped the coin three times and observed three heads.
4. If $A$ is a square matrix and $m$ is a positive integer, then the matrix power $A^m$ is defined inductively by $A^1 = A$ and $A^{m+1} = A^m \cdot A$. Consider the matrix $A$ defined by

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

(a) (5 points) Use the definition of matrix power and matrix product to calculate $A^m$ for $m = 1, 2, 3$.

(b) (10 points) Conjecture and prove (by mathematical induction) formulas for the entries of $A^m$ as functions of $m$ for the given matrix $A$.

5. (15 points) Find a basis for the vector space spanned by the vectors in the following set.

$$\{(6, 9, 3), (9, 14, 5), (-3, -4, -1)\}.$$ Justify your claim that it is a basis.

6. (20 points) Find the inverse of the following matrix $A$, showing the steps of Gauss-Jordan elimination (described in Chapter 13). Use fraction arithmetic, not decimals.

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 6 & 9 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$