1. (a) True.
Proof: Let $a$, $b$, $c$ be arbitrary integers.
Assume $a$ divides $b$ and $a$ divides $c$.
Then there exist integers $d_1$ and $d_2$ such that $b = d_1 \cdot a$ and $c = d_2 \cdot a$.
Then $b + c = d_1 \cdot a + d_2 \cdot a = (d_1 + d_2) \cdot a$
so $a$ divides $b + c$.
And $b \cdot c = d_1 \cdot a \cdot d_2 \cdot a = (d_1 \cdot a \cdot d_2) \cdot a$
so $a$ divides $b \cdot c$.
Since $a$, $b$, and $c$ were arbitrary integers, the implication holds for all integers $a$, $b$, and $c$.

1 (b) True.
Proof: Let $a$, $b$, $c$ be arbitrary integers.
Assume $a$ divides $b$ and $b$ divides $c$.
Then there exist integers $d_1$ and $d_2$ such that $b = d_1 \cdot a$ and $c = d_2 \cdot b$.
Thus $c = d_2 \cdot (d_1 \cdot a) = (d_2 \cdot d_1) \cdot a$
so $a$ divides $c$.
Because $a$, $b$, and $c$ were arbitrary integers, the implication holds for all integers $a$, $b$, and $c$.

1 (c) False
Counterexample:
Consider $a = 6$, $b = 2$, and $c = 3$.
Clearly $a$ divides $b \cdot c$, that is, $6$ divides $2 \cdot 3$, because $6 = 1 \cdot 6$.
However, $6$ does not divide $2$ and $6$ does not divide $3$. Thus, the implication does not hold for all integers $a$, $b$, and $c$. 
2. \((m,n)\) is defined to be \(\{\{m\},\{n\}\}\). We prove first that for all natural numbers \(a, b, c, d\) if \(a = c\) and \(b = d\) then \((a,b) = (c,d)\). Let \(a, b, c, d\) be arbitrary natural numbers. Assume \(a = c\) and \(b = d\). Then \(\{\{a\}\} = \{\{c\}\}\)
and \(\{\{a, b\}\} = \{\{c, d\}\}\). So \(\{\{\{a\}, \{a, b\}\}\} = \{\{\{c\}, \{c, d\}\}\}\).
That is \((a,b) = (c,d)\).

We next prove the converse, that for all natural numbers \(a, b, c, d\), if \((a,b) = (c,d)\) then \(a = c\) and \(b = d\).

Let \(a, b, c, d\) be arbitrary natural numbers. Assume \((a,b) = (c,d)\).
Then \(\{\{a\}\}, \{\{a, b\}\}\} = \{\{c\}, \{c, d\}\}\}.

We consider two cases:

Case (1) \(a = b\).
Then \(\{\{a, b\}\} = \{\{a\}\}, \) so \(\{\{\{a\}, \{a, b\}\}\} = \{\{\{a\}\}\}\}.
Because \(\{\{c\}, \{c, d\}\}\) \(\neq \{\{\{a\}\}\}\}.
It must be that the set on the left has only one element, that is \(\{\{c\}\} = \{c, d\}\). This means that \(c = d\), so \(\{\{c\}, \{c, d\}\} = \{\{c\}\}\). Thus, from \(\{\{\{c\}, \{c, d\}\}\} = \{\{\{a\}, \{a, b\}\}\}\}\}, we conclude that \(a = c\). And from \(a = b\) and \(c = d\), we have \(a = c\) and \(b = d\).)

Case (2) \(a \neq b\).
Then \(\{\{a\}, \{a, b\}\}\} contains two different sets, one with one element, and one with two elements. So \(\{\{c\}, \{c, d\}\}\} must contain two different sets, and \(c \neq d\). Thus, from \(\{\{a\}\} = \{\{c\}\}\} we get \(a = c\). Then, because
2. Case (2), continued
   \( a = c \) and \( \{a, b\} = \{c, d\} \), we must
   have \( b = d \).

   Thus in either case we have that
   for all natural numbers \( a, b, c, \) and
   \( d, (a, b) = (c, d) \) implies \( a = c \) and \( b = d \).

   Combining these two implications proves the
   claim.

3. (a)
   (i) \( \forall x \forall y \ f(x, y) = f(y, x) \)
   (ii) \( \forall e (N(e) \iff \forall x (f(e, x) = x)) \)
   (iii) \( \forall e_1 \forall e_2 (N(e_1) \land N(e_2) \rightarrow e_1 = e_2) \)
   (iv) \( \forall x \forall y (I(x, y) \iff N(f(x, y))) \)
   (v) \( \forall x \forall y \forall z (I(x, y) \land I(x, z) \rightarrow y = z) \)

2. (b)
   1. \( \forall x \forall y f(x, y) = f(y, x) \)  Promise (i)
   2. \( \forall e (N(e) \iff \forall x (f(e, x) = x)) \)  Promise (ii)
   3. \( N(e_1) \land N(e_2) \)  Assumption
   4. \( N(e_1) \)  Simplification (3)
   5. \( N(e_2) \)  Simplification (3)
   6. \( N(e_1) \iff \forall x (f(e_1, x) = x) \)  Universal Inst. (2)
   7. \( N(e_1) \rightarrow \forall x (f(e_1, x) = x) \)  Tautology (6)
   8. \( \forall x (f(e_1, x) = x) \)  \((p \Rightarrow q) \Rightarrow (p \Rightarrow q)\)
   9. \( f(e_1, e_1) = e_2 \)  Modus Ponens (4, 7)
   10. \( N(e_2) \iff \forall x (f(e_2, x) = x) \)  Universal Inst. (e)
   11. \( N(e_2) \rightarrow \forall x (f(e_2, x) = x) \)  Universal Inst. (2)
   12. \( \forall x (f(e_2, x) = x) \)  Tautology (10)(as for 7)
   13. \( f(e_2, e_1) = e_1 \)  Modus Ponens (5, 11)

(Continued \( \rightarrow \))
\[ e_2 \text{ for } x \]
\[ e_1 \text{ for } y \]

14. \[ f(e_2, e_1) = f(e_1, e_2) \quad \text{Universal Inst. (1)} \]

15. \[ f(e_2, e_1) = e_2 \quad \text{Substitution (9, 14)} \]

16. \[ e_1 = e_2 \quad \text{Substitution (13, 15)} \]

17. \[ N(e_1) \land N(e_2) \Rightarrow (e_1 = e_2) \quad \text{Deduction Theorem (3, 16)} \]

18. \[ \forall e_2 \left( N(e_1) \land N(e_2) \Rightarrow (e_1 = e_2) \right) \quad \text{Universal Generalization (17)} \]

19. \[ \forall e_1, \forall e_2 \left( N(e_1) \land N(e_2) \right) \quad \text{Universal Generalization (18)} \]

See next page for \( \exists e_3 \).
3(c) To see that (v) is not provable from (i)-(iv), we exhibit a situation in which (i)-(iv) are true, but (v) is false.

For example, let the set $S$ of elements be $\{a, b, c\}$ and let the function $f : S \times S \rightarrow S$ be given by the table

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>1</td>
</tr>
</tbody>
</table>

where $1 \neq a$

Define $N(x) = \begin{cases} 
\text{True} & \text{if } x = 1 \\
\text{False} & \text{otherwise} 
\end{cases}$

$I(x, y) = \begin{cases} 
\text{True} & \text{if } (x, y) \in \{(1,1), (a, a), (a, b), (b, a), (b, b)\} \\
\text{False} & \text{otherwise} 
\end{cases}$

Then continuing,

(i) For all $9$ pairs $(x, y) \in S \times S$, we have $f(x, y) = f(y, x)$. This is trivial if $x = y$, and

$f(1, a) = a = f(a, 1)$, $f(1, b) = b = f(b, 1)$ and

$f(a, b) = 1 = f(b, a)$

(ii) For $e = 1$, $N(e) = \text{True}$ and $f(1, 1) = 1$

$f(1, a) = a$

$f(1, b) = b$

For $e = a$, $N(e) = \text{False}$ and $f(a, a) = 1 \neq a$

$c = b$ $N(c) = \text{False}$ and $f(b, b) = 1 \neq b$

So, (ii) holds for all $e \in S$

(iii) We showed that if (i) & (ii) are true, then (iii) must also be true in 3(b)
(iv) For all pairs \((x, y) \in S \times S\), we have
\[
\begin{array}{ccc}
  x & y & f(x, y) \quad \Leftrightarrow \quad I(x, y) \\
  1 & 1 & \text{True} \\
  1 & a & a \\
  1 & b & b \\
  a & 1 & a \\
  a & a & 1 \\
  a & b & b \\
  b & 1 & b \\
  b & a & a \\
  b & b & 1
\end{array}
\]

So for all \((x, y)\), \(f(x, y)\) is a null element iff \(I(x, y)\) is true, so

(iv) is true.

But

(v) is false:

We have \(I(a, a)\) is true
\(I(a, b)\) is true
so \(a\) is an inverse of \(a\)
\(a\) is an inverse of \(b\)

But \(a \neq b\)
This gives a counterexample to (v)
when \(x = a\), \(y = a\), and \(z = b\),
so (v) is false

Hence (v) is not provable from (i) - (iv)
4, (a)

(i) \( \forall m \forall n \ (\neg (m > n) \iff (m \leq n)) \)
(ii) \( \forall n \ (C(n) \iff \exists m \ (\neg(m = 1) \land \\
\neg(m = n) \land D(m, n))) \)
(iii) \( \forall n \ (P(n) \iff (n > 1) \land \neg C(n)) \)
(iv) \( \forall n \ (n > 1) \rightarrow \exists p \ (P(p) \land D(p, n)) \)
(v) \( \forall n \ \forall d \ ((d \leq n) \rightarrow D(d, f(d))) \)
(vi) \( \forall n \ (\exists(f(n)) > 1) \)
(vii) \( \forall n \ \forall d \ ((d > 1) \rightarrow \neg(D(d, n) \land D(d, s(n)))) \)
(viii) \( \forall n \ \exists p \ (P(p) \land (p > n)) \)

4(b) Proof. Let 1 (i) - (vii) be premises (i) - (vii) above.

8. \( \exists n \ \forall p \ (P(p) \rightarrow \neg(P(p) > n)) \) Assumption (Negation of (viii)).
9. \( \forall n \ (P(p) \rightarrow \neg(P(p) > n)) \) Existential Instantiation (8).

10. \( s(f(n)) > 1 \) Universal Instantiation (6).
11. \( (s(f(n)) > 1) \rightarrow \exists p \ (P(p) \land D(p, s(f(n)))) \) U.I. (4).
12. \( \exists p \ (P(p) \land D(p, s(f(n)))) \) Modus Ponens (10, 11).
13. \( P(p) \land D(p, s(f(n))) \) Existential Instantiation (12).

14. \( P(p) \) Simplification (13).
15. \( D(p, s(f(n))) \) Simplification (13).

Po for p.
16. \( P(p) \rightarrow \neg(P(p) > n) \) U.I. (9).
17. \( \neg(P(p) > n) \) Modus Ponens (14, 16).

(continued ->)
(4(b) Continued)

po for n

18. \( \neg (p_0 > n_0) \leftrightarrow (p_0 \leq n_0) \)  
U.I. (1)

no for n

19. \( \neg (p_0 > n_0) \rightarrow (p_0 \leq n_0) \)  
Tautology (18)

\( (p \rightarrow q) \rightarrow (p \rightarrow q) \)

20. \( (p_0 \leq n_0) \)
Modus Ponens (17, 19)

\( \neg (p_0 \leq n_0) \rightarrow D(p_0, f(n_0)) \)
U.I. (5)

21. \( D(p_0, f(n_0)) \)
Modus Ponens (20, 21)

po for n

22. \( p(p_0) \leftrightarrow (p_0 > 1) \land \neg C(p_0) \)  
U.I. (3)

23. \( p(p_0) \rightarrow (p_0 > 1) \land \neg C(p_0) \)  
Tautology (23)

\( \text{(as for 19, )} \)

24. \( (p_0 > 1) \land \neg C(p_0) \)  
Modus Ponens (17, 24)

\( \neg (p_0 > 1) \rightarrow D(p_0, f(n_0)) \)
Simplification (25)

25. \( \neg (p_0 > 1) \leftrightarrow (D(p_0, f(n_0)) \land D(p_0, s(f(n_0)))) \)  
U.I. (17)

po for d

26. \( p_0 > 1 \)  
Simplification (25)

f(no) for n

27. \( p_0 > 1 \rightarrow \neg (D(p_0, f(n_0)) \land D(p_0, s(f(n_0)))) \)  
U.I. (17)

28. \( \neg (D(p_0, f(n_0)) \land D(p_0, s(f(n_0)))) \)  
Modus Ponens (26, 27)

29. \( D(p_0, f(n_0)) \lor D(p_0, s(f(n_0))) \)  
Conjunction (22, 15)

30. \( 0 \)  
Tautology (28, 29)

\( p \rightarrow 0 \)

We have proved the negation
of the assumption by contradiction, 
Hence, \( \text{(viii)} \).