Closed book and closed notes. No electronic devices. Show all written work on the test itself, using the backs of pages as necessary.

For problems that do not ask you to justify the answer, an answer alone is sufficient. However, if the answer is wrong and no derivation or supporting reasoning is given, there will be no partial credit.

This is a two-and-a-half hour exam, with an additional half hour “for improving what has already been written.” It will be collected at 5 pm.

GOOD LUCK!

<table>
<thead>
<tr>
<th>problem</th>
<th>possible</th>
<th>score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (10 points) Translate each of the following statements into a predicate logic formula. You may use functions \( \cdot \) and \(+\), and predicates \( =\), \(<\) as well as the constants 0 and 1 (but no other constants.) Assume that the domain is all integers.

(a) For every number \( a \), the sum of \( a \) and zero is equal to \( a \).

(b) For every number \( a \) there exists a number \( b \) such that the sum of \( a \) and \( b \) is equal to zero.

(c) The product of any two numbers less than zero is greater than zero.

(d) If \( a \) is not equal to zero and \( a \) times \( b \) is equal to \( a \) times \( c \), then \( b \) is equal to \( c \).

(e) If \( a \) divides \( b \) and \( b \) divides \( c \), then \( a \) divides \( c \).
2. (10 points) Give succinct definitions of the following terms. Assume $A$ and $B$ are sets and $f : A \rightarrow B$ is a function.

(a) The function $f$ is surjective.

(b) The Cartesian product of the sets $A$ and $B$, denoted $A \times B$.

(c) The set difference of the sets $A$ and $B$, denoted $A \setminus B$.

(d) The inverse function $f^{-1}$ (assuming that $f$ is a bijection.)

(e) The power set of the set $A$, denoted $\mathcal{P}(A)$. 

3. (10 points) Prove by simple mathematical induction that $3^n > 4n + 3$ for all integers $n \geq 3$. Please identify the predicate $P(n)$, the base case(s), and the induction hypothesis.
4. (a) (5 points) Find integers $s$ and $t$ such that $2 = 18s + 44t$, using the Euclidean algorithm for the gcd.

(b) (5 points) Let $S = \{0, 1, 2, \ldots, 43\}$. For how many integers $a \in S$ does there exist an integer $x \in S$ such that

$$ax \equiv 1 \pmod{44}.$$ 

Please justify your answer.
5. Let $F$ be the set of all functions $f(n)$ with domain and co-domain $\mathbb{N}$, where $\mathbb{N}$ is the set of all nonnegative integers.

(a) (4 points) Let $f(n), g(n) \in F$. Give the definition of: $f(n)$ is in $\Omega(g(n))$.

(b) (4 points) Prove that for all $f(n), g(n) \in F$, if $f(n)$ is in $O(g(n))$ then $g(n)$ is in $\Omega(f(n))$.

(c) (2 points) Prove or disprove: for all $f(n), g(n), h(n) \in F$, if $f(n)$ is in $O(h(n))$ and $g(n)$ is in $O(h(n))$ then $f(n) \cdot g(n)$ is in $O(h(n))$. 
6. (a) (5 points) Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be simple undirected graphs. We say that $H$ is a reduction of $G$ if and only if there exists a surjective function $g : V_G \to V_H$ such that for all edges $uv \in E_G$, we have $g(u)g(v) \in E_H$. Recall that $C_n$ is the undirected cycle graph of $n$ vertices and $n$ edges. Prove that $C_3$ is a reduction of $C_6$ and that $C_6$ is not a reduction of $C_3$.

(b) (5 points) Prove or disprove: if $G$ and $H$ are undirected graphs such that $G$ is a reduction of $H$ and $H$ is a reduction of $G$, then $G$ and $H$ are isomorphic, that is, there exists a bijection $f : V_G \to V_H$ such that for all $u, v \in V_G$, $uv \in E_G$ if and only if $f(u)f(v) \in E_H$. 
7. (a) (3 points) State the finite version of the Binomial Theorem, which gives an expression for the value of \((x + y)^n\).

(b) Consider the following identity:

\[ \sum_{k=1}^{n} \binom{n}{k} k = n2^{n-1}. \]

i. (2 points) Verify this identity for \(n = 5\), showing your work.

ii. (5 points) Prove this identity is true for all integers \(n \geq 1\).
8. Suppose there are two boxes: $A$ containing 6 balls (3 red, 2 yellow and 1 green), and $B$ containing 5 balls (2 red, 2 yellow and 1 green). I select one of the two boxes uniformly at random (noting which one) and then select from that box one of the balls it contains, again uniformly at random (noting its color). For example, if I choose box $A$ and choose a red ball from that box, then the outcome is $(A, R)$.

(a) (3 points) Give the 6 possible outcomes of this experiment and their probabilities.

(b) (4 points) If $A$ is the event that box $A$ is chosen and $R$ is the event that a red ball is chosen, calculate the probabilities $\Pr(A)$, $\Pr(R)$, $\Pr(A \cap R)$ and $\Pr(A|R)$.

(c) (3 points) In this probability space, we define the random variable $X$ to have value 1 if a red ball is chosen, 2 if a yellow ball is chosen, and 3 if a green ball is chosen. Find the expected value of $X$, showing you calculation.
9. (a) (4 points) Let \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) be a set of vectors from \( \mathbb{R}^n \). Give definitions for: (i) \( \mathbf{v} \) is a linear combination of vectors from \( S \) and (ii) the span of \( S \).

(b) (6 points) Let \( A \) be the matrix given by:

\[
A = \begin{pmatrix}
1 & 1 & 2 \\
2 & 2 & 3 \\
3 & 2 & 3
\end{pmatrix}
\]

Compute the inverse of \( A \) using Gauss-Jordan elimination, indicating the (one or two) row operations performed in each step. Indicate what \( A^{-1} \) is.
10. (a) (4 points) Draw the simple undirected graph $G$ whose vertices are $\{1, 2, 3, 4\}$ and whose adjacency matrix $A$ is the following.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Is $G$ a tree? Why or why not?

(b) 6 points Suppose we randomly generate a simple undirected graph $H$ on the vertices $\{1, 2, 3, 4, 5\}$ by choosing, for each possible edge $uv$, to include $uv$ with probability $1/2$, independent of all other choices. What is the probability the resulting graph $H$ is a tree? (You must justify your answer.)