PART I

1. (15 points)
Consider the outcome space $\Omega$ of all strings of length 4 over the alphabet \{a, b, c, d, e\} that contain no repeated letters. Thus, $acbe \in \Omega$, but $baca \notin \Omega$. The probability function $\Pr : \Omega \rightarrow [0, 1]$ assigns the same probability to every element of $\Omega$. For each $i = 1, 2, 3, 4$, let $X_i$ be the value of the $i$th letter, numbering from the left. Please give justifications for your answers to the following.

(a) How many elements are there in $\Omega$, and what is the probability of each one?

(b) Let $A$ be the event that $X_1 = a$ and $B$ be the event that $X_2 = b$. What are $\Pr(A)$ and $\Pr(A \cap B)$?

(c) For the events $A$ and $B$ defined in part (b), what is $\Pr(B \mid A)$? Are the events $A$ and $B$ independent?

(d) For every pair $(i, j)$ of integers such that $1 \leq i < j \leq 4$, define the random variable $Y_{(i,j)}$ to be 1 if letter $X_i$ alphabetically precedes letter $X_j$, or 0 otherwise. What is $\Pr(Y_{(1,4)} = 1)$?

(e) Let the random variable $Y$ be the sum of the all the random variables $Y_{(i,j)}$ defined in part (d). For every value that $Y$ can take on, give an example of an element of $\Omega$ for which it takes that value. What is $E(Y)$?

2. (15 points) Let $X$ be a binomially distributed random variable with $p = 0.7$ and $n = 3$.

(a) Give a table of the values (as decimal fractions) of the probability distribution for $X$.

(b) Which value of $X$ has the highest probability? Is it more probable that $X$ will be even or more probable that $X$ will be odd?
(c) What are the expectation and variance of $X$? (See Chapter 12 for definitions.)

3. Suppose there are three identical looking coins: $C_i$ for $i = 1, 2, 3$, with probability of heads of 0.25, 0.50, and 0.75, respectively. I cannot tell the coins apart, but I select one of the three coins uniformly at random and hand it to someone, who flips it three times and reports that there were two heads and a tail, (but not the order in which these occurred.)

(a) (5 points) Describe an outcome space $\Omega$ and probability function $\Pr : \Omega \rightarrow [0, 1]$ to represent this experiment. You should be able to represent the event $A_i$ that I chose coin $C_i$ (for $i = 1, 2, 3$) and the event $B$ that three flips yielded two heads and one tail.

(b) (5 points) What is the probability of the event $B$?

(c) (10 points) Because I select a coin equiprobably, we have $\Pr(A_i) = 1/3$ for $i = 1, 2, 3$. Calculate $\Pr(A_i \mid B)$ for $i = 1, 2, 3$. (Hint: Bayes’ Theorem might help.)

PART II

4. (15 points) Find the inverse of the following matrix $A$, showing the steps of Gauss-Jordan elimination (described in Chapter 13). Use fraction arithmetic, not decimals.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

5. (15 points) Use matrix identities (from 13.3.3) to show that if $A$ is an $n \times n$ dimensional matrix that is invertible, then the function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $f(\vec{x}) = A \cdot \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ is bijective.

6. Let the matrix $A$ be given by

$$A = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Let $\vec{x} = (9, 4, 1)^T$ be a column vector. (It is written as a row vector, with an indication it should be transposed to get a column vector.)

(a) (10 points) Compute $A \cdot \vec{x}$ and $A^2 \cdot \vec{x}$.

(b) (10 points) Conjecture and prove a formula giving $A^n \cdot \vec{x}$ in terms of $n$, for all $n \in \mathbb{N}$. 