PART I

1. (15 points) For integers \(a\) and \(b\), we say that \(a\) divides \(b\) if and only if there exists an integer \(d\) such that \(b = d \cdot a\). Express each of the following false statements about the domain of integers as a closed predicate logic formula, and give a counterexample to show that it is false. Use the binary predicates \(\text{divides}(a, b)\) for “\(a\) divides \(b\)” and \((a = b)\) for “\(a\) is equal to \(b\)” and the binary function \((a \cdot b)\) for the product of \(a\) and \(b\).

   (a) For all integers \(a\) and \(b\), if \(a\) divides \(b\) and \(b\) divides \(a\) then \(a\) and \(b\) are equal.

   (b) For all integers \(a\), \(b\), and \(c\), if \(a\) divides \(c\) and \(b\) divides \(c\) then the product of \(a\) and \(b\) divides \(c\).

   (c) For every integer \(a\) there is an integer \(b\) which is not equal to \(a\) such that \(a\) divides \(b\).

2. (15 points) For each of the following statements about the integers, give its negation in a form in which any occurrence of not (\(\neg\)) is applied directly to an atomic formula, i.e., to a predicate applied to terms. De Morgan’s Laws and their generalization to quantified formulas will be helpful. Say which of the two (the original formula or its negation) is true of the integers and justify your answer with an argument. These formulas use the binary predicates \((x \leq y)\) for “\(x\) is less than or equal to \(y\)” and \((x = y)\) for “\(x\) is equal to \(y\)” and the binary function \((x + y)\) for the sum of \(x\) and \(y\), as well as the constant 0. Do not introduce any additional predicates, functions or constants.

   (a) \(\forall x)(\exists y)((x + y) = 0)\)

   (b) \(\forall x)(\forall y)(\forall z)((x \leq y) \to ((z \cdot x) \leq (z \cdot y)))\)

   (c) \(\forall x)(\exists y)((\neg(y \leq x)) \land (\forall z)((z \leq x) \lor (y \leq z)))\)

3. (20 points) Let \(A\), \(B\), and \(C\) be unary predicates. For each of the following “maybe syllogisms”, determine if the third line is a logical consequence of the first two. If so, give an informal argument to show that it is a logical consequence. If not, give an example of a domain (set of elements) and definitions of the predicates \(A\), \(B\), and \(C\) for which the first two lines are true but the third is false.

PART II
(a)  i. $(\forall x)(A(x) \rightarrow B(x))$
    ii. $(\forall x)(B(x) \rightarrow C(x))$
    iii. $(\forall x)(A(x) \rightarrow C(x))$

(b)  i. $(\forall x)(A(x) \rightarrow B(x))$
    ii. $(\exists x)((\neg A(x)) \land C(x))$
    iii. $(\exists x)((\neg B(x)) \land C(x))$

(c)  i. $(\forall x)(A(x) \rightarrow B(x))$
    ii. $(\exists x)((\neg B(x)) \land C(x))$
    iii. $(\exists x)((\neg A(x)) \land C(x))$

(d)  i. $(\exists x)(A(x) \land B(x))$
    ii. $(\exists x)(B(x) \land C(x))$
    iii. $(\exists x)(A(x) \land C(x))$

PART II

4. (15 points) Consider the domain of integers. We define $n$ to be even if and only if there exists $m$ such that $n = 2m$. We define $n$ to be odd if and only if there exists $m$ such that $n = 2m + 1$. You may assume every integer is even or odd and no integer is both even and odd.

For each of the following statements, translate it into a closed predicate logic formula and then give an informal proof of it. In your predicate logic formulas, use the unary predicates $\text{even}(n)$ and $\text{odd}(n)$, and the binary function $(m \cdot n)$ for the product of $m$ and $n$. However, you may use general algebraic notation in your informal proofs.

(a) If $n$ is even then $n^2$ is even.
(b) If $n$ is odd then $n^2$ is odd.
(c) If $n^2$ is even then $n$ is even.

5. (15 points) Consider the domain of nonnegative integers, with the binary predicate $\text{divides}(a, b)$ defined to be true if and only if there exists a nonnegative integer $d$ such that $b = d \cdot a$. Define the unary predicate $\text{prime}(n)$ to be true if and only $n \neq 1$, and the only nonnegative integers that divide $n$ are 1 and $n$.

(a) Write a closed predicate logic formula giving the above definition of $\text{divides}(a, b)$, using the binary predicate $(x = y)$ for equality and the binary function $(x \cdot y)$ for the product of $x$ and $y$.

(b) In lecture we expressed the definition of $\text{prime}(n)$ as follows.

$$(\forall n)(\text{prime}(n) \leftrightarrow (\neg(n = 1) \land (\forall x)(\text{divides}(x, n) \rightarrow ((x = 1) \lor (x = n))))).$$
Define a new unary predicate \textbf{composite}(n) to be true if and only if \( n \neq 1 \) and \( n \) is not prime. Express the definition of \textbf{composite}(n) as a closed predicate logic formula without using the predicate \textbf{prime}.

(c) In lecture we expressed the statement “there is no largest prime” as follows.

\[(\forall n)(\text{prime}(n) \rightarrow (\exists x)(\text{prime}(x) \land (x > n)))\]

Is this logically equivalent to the following formula?

\[\neg(\exists n)((\text{prime}(n) \land (\forall x)(\text{prime}(x) \rightarrow (\neg(x > n)))))\]

Justify your answer.

6. (20 points)

(a) Prove that for any nonnegative integer \( m \), the only nonnegative integer that divides both \( m \) and \( m + 1 \) is 1.

(b) Assume as a lemma that if \( n \) is any composite nonnegative integer, then there is a prime that divides \( n \). Let \( p_1, p_2, \ldots, p_n \) be any nonempty list of primes. Prove that there is a prime \( q \) not on this list. (Hint: consider the number \( 1 + p_1 \cdot p_2 \cdot \cdots \cdot p_n \).

Why does this also prove that there is no largest prime number?