Please submit your homework in **two separate parts**: problems 1-3 in the first part, and 4-6 in the second part. Make sure your name is on both parts. In the first part, please include the names of anyone (including course staff) you consulted with in connection with this assignment, as well as listing any resources (including course materials) you consulted.

In each proof by simple or strong induction on integers, please explicitly state (and label) the predicate $P(n)$, a predicate logic formula using $P(n)$ that states the claim to be proved, the base case(s), and the inductive hypothesis.

**PART I**

1. (15 points) Prove using simple mathematical induction that for all positive integers $n$ the sum of the first $n$ odd positive integers is $n^2$.

   **Solution.** The predicate $P(n)$ is:
   $$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$  
   The claim to be proved is:
   $$(\forall n \in \mathbb{Z})( (n \geq 1) \rightarrow P(n) ).$$
   The base case is $P(1)$:
   $$1 = 1^2,$$  
   which is true. For the inductive step, we assume the inductive hypothesis $P(n)$:
   $$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$  
   is true for some integer $n \geq 1$. We add the next term, $(2n + 1)$, to both sides:
   $$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + 2n + 1,$$  
   and observe that $n^2 + 2n + 1 = (n + 1)^2$, to conclude that
   $$1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2,$$  
   which is is $P(n + 1)$. We have shown that for every positive integer $n$,
   $$1 + 3 + 5 + \ldots + (2n - 1) = n^2.$$

2. (15 points) Prove using simple mathematical induction that for all integers $n \geq 2$,
   $$6^n > 2^{2n+1}.$$  

   **Solution.** The predicate $P(n)$ is:
   $$6^n > 2^{2n+1}.$$  
   The claim to be proved is:
   $$(\forall n \in \mathbb{Z})( (n \geq 2) \rightarrow P(n) ).$$
The base case is $P(2)$:

$$6^2 = 36 > 32 = 2^5 = 2^{2+1},$$

which is true. The inductive hypothesis is $P(n)$ is true, that is,

$$6^n > 2^{2n+1}$$

is true for some integer $n \geq 2$. If we multiply both sides of this inequality by 6, the inequality is preserved:

$$6 \cdot 6^n > 6 \cdot 2^{2n+1}.$$ 

Because $6 > 4$, we know $6 \cdot 2^{2n+1} > 4 \cdot 2^{2n+1}$, so we have

$$6^{n+1} > 4 \cdot 2^{2n+1} = 2^{2n+3} = 2^{2(n+1)+1},$$

which is $P(n+1)$. Thus we have shown that for every integer $n \geq 2$, we have $6^n > 4^{2n+1}$.

3. (20 points) Conjecture a formula in terms of $n$ for the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)}$$

and prove its correctness using simple mathematical induction.

Solution. To save writing, we define the function $f(n)$ by

$$f(n) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n+1)},$$

and evaluate $f(n)$ for $n = 1, 2, 3, 4$ to get

$$f(1) = \frac{1}{2},$$

$$f(2) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

$$f(3) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4},$$

$$f(4) = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}.$$ 

The pattern appears to be that $f(n) = n/(n+1)$; we will prove this. The claim to be proved is that for all positive integers $n$, $f(n) = n/(n+1)$. The predicate $P(n)$ is:

$$f(n) = \frac{n}{n+1}.$$ 

Then the claim to be proved is

$$(\forall n \in \mathbb{Z})(n \geq 1) \rightarrow P(n).$$

The base case is $P(1)$:

$$f(1) = \frac{1}{1+1}.$$
which is true. (We calculated the value of \( f(1) \) above.) The induction hypothesis is that \( P(n) \) is true, that is,
\[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n + 1)} = \frac{n}{n + 1}, \]
is true for some integer \( n \geq 1 \). We add the next term, \( \frac{1}{(n + 1)(n + 2)} \), to both sides, add the two fractions on the right hand side, and simplify the result:
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n \cdot (n + 1)} + \frac{1}{(n + 1) \cdot (n + 2)} = \frac{n}{n + 1} + \frac{1}{(n + 1) \cdot (n + 2)} = \frac{(n \cdot (n + 2)) + 1}{(n + 1) \cdot (n + 2)} = \frac{n^2 + 2n + 1}{(n + 1) \cdot (n + 2)} = \frac{(n + 1) \cdot (n + 1)}{(n + 1) \cdot (n + 2)} = \frac{n + 1}{n + 2},
\]
which is \( P(n + 1) \). We have shown that for every positive integer \( n \), \( f(n) = n/(n + 1) \).

PART II

4. (15 points) Let \( S \) be the set of positive integers \( n \) such that a postage of \( n \) cents can be made using stamps of denominations 3 cents and/or 7 cents. Prove using strong mathematical induction that every integer \( n \geq 12 \) is an element of \( S \).

\textit{Solution.} The predicate \( P(n) \) is that \( n \in S \). The claim to be proved is:
\[
(\forall n \in \mathbb{Z})((n \geq 12) \rightarrow (n \in S)).
\]

We use three base cases, 12, 13, and 14. \( P(12) \) is true because \( 12 = 4 \cdot 3 \). \( P(13) \) is true because \( 13 = 2 \cdot 3 + 7 \). \( P(14) \) is true because \( 14 = 2 \cdot 7 \). The induction hypothesis is that
\[
P(12), P(13), P(14), \ldots, P(n)
\]
are all true, for some integer \( n \geq 14 \). Then \( n + 1 \geq 15 \), and subtracting 3 from both sides, \( n - 2 \geq 12 \). Then \( n - 2 \) is a positive integer greater than or equal to 12 and less than or equal to \( n \), so by the induction hypothesis, we know that \( P(n - 2) \) is true, that is, \( (n - 2) \in S \). Thus we can make the denomination \( (n - 2) \) using 3 and 7 cent stamps, and by adding another 3 cent stamp, we can make the denomination \( (n + 1) \), so \( (n + 1) \in S \). Thus, \( P(n + 1) \) is true.

We have shown that \( n \in S \) for every integer \( n \geq 12 \).

5. We give a recursive definition of an \textit{s-formula} as follows.

- The variables \( x, y, \) and \( z \) are \( s \)-formulas.
- If \( F \) and \( G \) are \( s \)-formulas, then \( (\neg F) \) and \( (F + G) \) are \( s \)-formulas.
(a) (5 points) If $F$ is an s-formula, we define $v(F)$ to be the number of occurrences of variables in $F$ and $s(F)$ to be the number of occurrences of $+$ signs in $F$. An example of an s-formula is $F = ((-x + y)) + ((y + z))$, for which $v(F) = 4$ and $s(F) = 3$. Give five more examples of s-formulas $F$ and the values of $v(F)$ and $s(F)$ for them.

Solution.

<table>
<thead>
<tr>
<th>$F$</th>
<th>$v(F)$</th>
<th>$s(F)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$(x + (-z))$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$((x + y) + ((y + x) + z))$</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>$(-(-(-x)))$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$(x + (y + z))$</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) (10 points) Prove by structural induction that for any s-formula $F$, $v(F) = s(F) + 1$.

Note that the predicate $P(F)$ is the statement that $v(F) = s(F) + 1$. Explicitly state (and label) your base case(s) and inductive hypothesis.

Solution. The base case is when $F$ is a single variable, $x$, $y$, or $z$. In this case, $v(F) = 1$ and $s(F) = 0$, so $v(F) = s(F) + 1$. The inductive hypothesis is that $F_1$ and $F_2$ are s-formulas and $v(F_1) = s(F_1) + 1$ and $v(F_2) = s(F_2) + 1$. There are two cases of $F$ to consider: (1) $F$ is $(-F_1)$ and (2) $F$ is $(F_1 + F_2)$. In the first case, $F$ has the same numbers of occurrences of variables and plus signs as $F_1$, so we have

$$v(F) = v(F_1) = s(F_1) + 1 = s(F) + 1,$$

using the induction hypothesis for $F_1$. In the second case, the number of occurrences of variables in $F$ is the sum of the number of occurrences of variables in $F_1$ and $F_2$. And the number of occurrences of $+$ signs in $F$ is the sum of the number of occurrences of $+$ signs in $F_1$ and $F_2$, plus 1. That is,

$$v(F) = v(F_1) + v(F_2),$$

and

$$s(F) = s(F_1) + s(F_2) + 1.$$

Using the inductive hypothesis for $F_1$ and $F_2$, we have

$$v(F) = v(F_1) + v(F_2) = s(F_1) + 1 + s(F_1) + 1 = s(F) + 1.$$

Thus, in either case we have $P(F)$ is true. We have proved that for every s-formula $F$, $v(F) = s(F) + 1$.

6. (20 points) For positive integers $r$ and $s$, define $(r \mod s)$ to be the remainder when we divide $r$ by $s$. We recursively define a procedure to compute a function $f$ on the domain of all positive integers as follows. The base cases are $f(1) = f(2) = f(3) = 1$, and if $n \geq 4$ then

$$f(n) = f(n/4) \text{ if } (n \mod 4) = 0$$
$$f(n) = f(3n + 1) \text{ if } (n \mod 4) = 1$$
$$f(n) = f(n + 2) \text{ if } (n \mod 4) = 2$$
$$f(n) = f(n + 3) \text{ if } (n \mod 4) = 3.$$
Prove using strong mathematical induction that this procedure halts and returns $f(n) = 1$ for all positive integers $n$.

Solution. The predicate $P(n)$ is that $f(n) = 1$. The claim to be proved is

$$(\forall n \in \mathbb{Z})(n \geq 1 \rightarrow P(n)).$$

The base case is $P(1)$: we know $f(1) = 1$ by the definition of $f$. The (strong) induction hypothesis is:

$P(1), P(2), \ldots, P(n)$ are all true, for some integer $n \geq 1$. Consider the next integer, $(n+1)$, which is at least 2. Let $q$ be the quotient when we divide $(n+1)$ by 4. There are four cases, depending on the remainder $(n+1) \mod 4$.

- **Case**: $(n+1) = 4q$. In this case, $f(n+1) = f(4q/4) = f(q)$. Because $q$ is a positive integer less than or equal to $n$, we know $P(q)$ is true (by the induction hypothesis), so $f(q) = 1$ and $f(n+1) = 1$.

- **Case**: $(n+1) = 4q + 1$. In this case, because $(n+1) \geq 2$, we know $q \geq 1$. Using the definition of $f$,

  $$f(n+1) = f(4q + 1) = f(3(4q + 1) + 1) = f(12q + 4).$$

  But $(12q + 4)$ is a multiple of 4, so we can apply the definition of $f$ again to get

  $$f(n+1) = f(3q + 1).$$

  Because $q \geq 1$, we know $(3q + 1) < (4q + 1)$, so $(3q + 1)$ is a positive integer less than $(n+1)$, and $P(3q+1)$ is true by the inductive hypothesis. Thus, $f(n+1) = f(3q+1) = 1$.

- **Case**: $(n+1) = 4q + 2$. In this case, using the definition of $f$,

  $$f(n+1) = f(4q + 2) = f(4q + 4).$$

  Using the definition of $f$ for a multiple of 4,

  $$f(n+1) = f(q+1).$$

  But $q \geq 0$ and $(q+1) < (4q + 2)$, so $q+1$ is a positive integer less than $(n+1)$, and $P(q+1)$ is true by the inductive hypothesis. Thus, $f(n+1) = f(q+1) = 1$.

- **Case**: $(n+1) = 4q + 3$. In this case, using the definition of $f$,

  $$f(n+1) = f(4q + 3) = f(4q + 6) = f(4(q + 1) + 2).$$

  Using the definition of $f$ again (for remainder 2),

  $$f(n+1) = f(4(q + 1) + 4) = f(4(q + 2)).$$

  Using the definition of $f$ yet again (for remainder 0),

  $$f(n+1) = f(q + 2).$$

  Then because $q \geq 0$, $(q + 2) < (4q + 3)$ and $(q + 2)$ is a positive integer that is less than $(n+1)$ and $P(q+2)$ is true by the inductive hypothesis. Thus, $f(n+1) = f(q + 2) = 1$.

Thus, in all four cases, $f(n+1) = 1$, which is the statement $P(n+1)$. We have proved that for all positive integers $n$, $f(n) = 1$.