1. (15 points) For each of the following binary relations $R$ on the set $A$, determine whether it has each of the following properties, and briefly justify your answer.

Relations:

(a) $A$ is the positive integers and $R$ is the set of pairs $(m, n)$ such that every prime that divides $m$ also divides $n$.

Solution. $R$ is reflexive: for every positive integer $m$, every prime that divides $m$ also divides $m$, so $(m, m) \in R$. $R$ is not symmetric: every prime that divides 2 also divides 6, but not vice versa, so $(2, 6) \in R$ but $(6, 2) \notin R$. $R$ is not antisymmetric: every prime that divides 6 also divides 12 and vice versa, so $(6, 12) \in R$ and $(12, 6) \in R$, but $6 \neq 12$. $R$ is transitive: suppose $(m, n) \in R$ and $(n, s) \in R$. Then if $p$ is a prime that divides $m$, it also divides $n$ and therefore also divides $s$, so every prime that divides $m$ also divides $s$ and $(m, s) \in R$. $R$ is not an equivalence relation: although it is reflexive and transitive, it is not symmetric.

(b) $A$ is all subsets of the nonnegative integers, i.e., $A = \mathcal{P}(\mathbb{N})$, and $R$ is the set of pairs $(S, T)$ such that there exists a bijection $f : S \to T$.

Solution. $R$ is reflexive: if $S \subseteq \mathbb{N}$, then we may define a bijection $f : S \to S$ by $f(n) = n$ for all $n \in S$, so $(S, S) \in R$. $R$ is symmetric: if $(S, T) \in R$, then there exists a bijection $f : S \to T$. Consider its inverse $f^{-1} : T \to S$. This is a bijection mapping $T$ to $S$, so $(T, S) \in R$. $R$ is not antisymmetric: we can define a bijection from $\{1\}$ to $\{2\}$ by $f(1) = 2$, and a bijection $g$ from $\{2\}$ to $\{1\}$ by $g(2) = 1$, so $((\{1\}, \{2\}) \in R$ and $(\{2\}, \{1\}) \in R$, but clearly $\{1\} \neq \{2\}$ because $1 \neq 2$. $R$ is transitive: suppose $(S, T) \in R$ and $(T, U) \in R$. Then there exist bijections $f : S \to T$ and $g : T \to U$. We claim their composition, $(g \circ f)$ is a bijection from $S$ to $U$. If $s_1, s_2 \in S$ and $s_1 \neq s_2$, then because $f$ is injective, $f(s_1) \neq f(s_2)$, and because $g$ is injective, $g(f(s_1)) \neq g(f(s_2))$, so $(g \circ f)$ is injective. If $u \in U$, then because $g$ is surjective, there exists $t \in T$ such that $g(t) = u$, and because $f$ is surjective, there exists $s \in S$ such that $f(s) = t$, so $g(f(s)) = u$, and $(g \circ f)$ is surjective, so $(g \circ f)$ is a bijection. $R$ is an equivalence relation: it is reflexive, symmetric, and transitive.

(c) $A$ is the set $\mathbb{N}$ of all nonnegative integers and $R$ is the set of pairs $(m, n)$ such that $m^2$ divides $n$.

Solution. $R$ is not reflexive: 4 does not divide 2, so $(2, 2) \notin R$. $R$ is not symmetric: $(2, 4) \in R$ but $(4, 2) \notin R$. $R$ is antisymmetric: suppose that $(m, n) \in R$ and $(n, m) \in R$. This means that $m^2$ divides $n$ and $n^2$ divides $m$. Thus, $m^2 \leq n$ and $n^2 \leq m$. We also have that $m \leq m^2$, so $m \leq n$, and $n \leq n^2$, so $n \leq m$. Because $m \leq n$ and $n \leq m$, we have $m = n$. $R$ is transitive: if $(m, n) \in R$ and $(n, s) \in R$, then we know $m^2$ divides $n$
and $n^2$ divides $s$. Because the divides relation is transitive and $n$ divides $n^2$, this means that $m^2$ divides $n^2$ and therefore $m^2$ divides $s$, so $(m, s) \in R$. $R$ is not an equivalence relation: although it is transitive, it is neither reflexive nor symmetric.

Properties:

i. $R$ is reflexive.

ii. $R$ is symmetric.

iii. $R$ is antisymmetric.

iv. $R$ is transitive.

v. $R$ is an equivalence relation.

2. (15 points) For each of the following sets of properties, determine whether there exists a binary relation $R$ on the set \{a, b, c, d\} that has the given set of properties. If so, give such a relation as a directed graph (not a Hasse diagram), and explain why the stated properties are true of the relation. If not, prove that no such relation exists.

(a) $R$ is reflexive, symmetric, and not transitive.

Solution.

\[ a \xleftarrow{} b \quad b \xrightarrow{} c \quad c \xrightarrow{} a \quad d \xrightarrow{} c \]

(Diagram of $R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (b, c), (c, b)\}$.) $R$ is reflexive: there are self-loops on $a$, $b$, $c$, and $d$. $R$ is symmetric: every edge $(x, y)$ has a reverse edge $(y, x)$. $R$ is not transitive: $(a, b) \in R$ and $(b, c) \in R$, but $(a, c) \notin R$.

(b) $R$ is not reflexive, antisymmetric, and not transitive.

Solution.

\[ a \xrightarrow{} b \quad b \xrightarrow{} c \quad d \xrightarrow{} c \]

(Diagram of $R = \{(a, b), (b, c)\}$.) $R$ is not reflexive: $(a, a) \notin R$. $R$ is antisymmetric: there are no pairs of edges $(x, y)$ and $(y, x)$ with $x \neq y$. $R$ is not transitive: $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$.

(c) $R$ is not reflexive, not symmetric, not antisymmetric, and transitive.

Solution.

\[ a \xrightarrow{} b \quad b \xleftarrow{} c \quad d \xrightarrow{} c \]

(Diagram of $R = \{(a, b), (a, c), (b, b), (b, c), (c, b), (c, c)\}$.) $R$ is not reflexive: $(a, a) \notin R$. $R$ is not symmetric: $(a, b) \in R$ but $(b, a) \notin R$. $R$ is not antisymmetric: $(b, c) \in R$ and $(c, b) \in R$ but $b \neq c$. $R$ is transitive: if $(x, y) \in R$ and $(y, z) \in R$ then either $y = b$ or $y = c$. If $y = b$ then $x$ is $a$, $b$ or $c$ and $z$ is $b$ or $c$, and all the edges $(a, b), (a, c), (b, b), (b, c), (c, b)$ and $(c, c)$ are present. Similarly for $y = c$.  

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(d) $R$ is a partial order with one minimum element and one maximum element, but $R$ is not a total order.

Solution.

(Diagram of $R = \{(a,a), (a,b), (b,b), (b,d), (c,c), (c,d), (d,d)\}$.) $R$ is reflexive: there are self-loops on all four nodes: $a$, $b$, $c$, and $d$. $R$ is antisymmetric: there is no pair of edges $(x,y) \in R$ and $(y,x) \in R$ with $x \neq y$. $R$ is transitive: because it is reflexive, we need only consider the cases: (1) $(a,b) \in R$ and $(b,d) \in R$, and also $(a,d) \in R$ and (2) $(a,c) \in R$ and $(c,d) \in R$, and also $(a,d) \in R$. Thus, $R$ is a partial order. Because $b$ and $c$ are not comparable, $R$ is not a total order. Because $(a,b)$, $(a,c)$ and $(a,d)$ are all in $R$, $a$ is the unique minimum element, and because $(a,d)$, $(b,d)$ and $(c,d)$ are all in $R$, $d$ is the unique maximum element.

(e) $R$ is partial order with two minimal elements and two maximal elements.

Solution.

(Diagram of $R = \{(a,a), (a,b), (b,b), (b,d), (c,c), (c,d), (d,d)\}$.) $R$ is reflexive: each of the four nodes $a$, $b$, $c$ and $d$ has a self-loop. $R$ is antisymmetric: there is no pair of edges $(x,y)$ and $(y,x)$ such that $x \neq y$. $R$ is transitive: if $(x,y) \in R$ and $(y,z) \in R$, then at least one of the two must be a self-loop, which means $(x,z) \in R$. Thus, $R$ is a partial order. Because there is no element $x$ such that $(x,a)$ and $x \neq a$, $a$ is a minimal element, and similarly for $c$. Because there is no element $y$ such that $(b,y)$ and $y \neq b$, $b$ is a maximal element, and similarly for $d$. The elements $a$ and $c$ are the only minimal elements, because for $b$ we have $(a,b) \in R$ and for $d$, we have $(c,d) \in R$. Analogously, $b$ and $d$ are the only maximal elements.

3. (20 points) Let $R$ be a binary relation on a set $A$. Recall that $R \circ R$ is the binary relation on $A$ consisting of the set of all pairs $(x,z)$ such that for some $y \in A$, $(x,y) \in R$ and $(y,z) \in R$.

Prove or disprove each of the following statements:

(a) If $R$ is an equivalence relation, then $R \circ R$ is an equivalence relation.

Solution. We prove this is true. Let $R$ be an equivalence relation on $A$. To see that $(R \circ R)$ is reflexive, let $a \in A$. Because $R$ is reflexive, $(a,a) \in R$, so $(a,a) \in (R \circ R)$ (letting $x = y = z = a$ in the definition of $(R \circ R)$.) Thus, $(R \circ R)$ is reflexive. To see that $(R \circ R)$ is symmetric, let $(a,b) \in (R \circ R)$. By definition of $(R \circ R)$, there exists $c \in A$ such that $(a,c) \in R$ and $(c,b) \in R$. Because $R$ is symmetric, this implies that $(c,a) \in R$ and $(b,c) \in R$. By the definition of $(R \circ R)$, this implies that $(b,a) \in (R \circ R)$ (with $x = b$, $y = c$ and $z = a$). Thus, $(R \circ R)$ is symmetric. To see that $(R \circ R)$ is
transitive, assume that \((a, b)\) and \((b, c)\) are in \((R \circ R)\). Then there exist \(y_1\) and \(y_2\) in \(A\) such that \((a, y_1) \in R\), \((y_1, b) \in R\), \((y_2, b) \in R\) and \((y_2, c) \in R\). Then, because \(R\) is transitive, we have \((a, b) \in R\), and \((b, c) \in R\). By the definition of \((R \circ R)\), this implies that \((a, c) \in (R \circ R)\) (with \(x = a\), \(y = b\), and \(z = c\)). Thus, \((R \circ R)\) is transitive. Because \((R \circ R)\) is reflexive, symmetric, and transitive, we conclude that \((R \circ R)\) is an equivalence relation.

(b) If \(R\) is a partial order, then \(R \circ R\) is a partial order.

Solution. We prove this is true. Assume that \(R\) is a partial order on \(A\). Then \(R\) is reflexive, and we showed in part (a) that if \(R\) is reflexive, then \((R \circ R)\) is reflexive, so \((R \circ R)\) is reflexive. To see that \((R \circ R)\) is antisymmetric, assume \((a, b)\) and \((b, a)\) are in \((R \circ R)\). Then there exist \(y_1\) and \(y_2\) in \(A\) such that \((a, y_1) \in R\), \((y_1, b) \in R\), \((b, y_2) \in R\) and \((y_2, a) \in R\). Because \(R\) is a partial order, it is transitive, so this implies that \((a, b) \in R\) and \((b, a) \in R\). Because \(R\) is a partial order, and therefore antisymmetric, this implies that \(a = b\). Thus, \((R \circ R)\) is antisymmetric. Because \(R\) is a partial order, it is transitive, and we proved in part (a) that if \(R\) is transitive, then \((R \circ R)\) is transitive, so \((R \circ R)\) is transitive. Because \((R \circ R)\) is reflexive, antisymmetric and transitive, it is a partial order on \(A\).

4. (15 points) Let \(A\) be the set of all nonempty finite strings of lower-case letters of the English alphabet. Thus, \(A\) contains strings \(app, apple, zzyxv\), among many others. Define the relation \(R\) on \(A\) to be the set of all pairs of strings \((s, t)\) such that either \(s = t\) or, if they are different, then \(s\) would precede \(t\) in a dictionary, with the usual alphabetic ordering of the letters. Thus, \((app, apple) \in R\), and \((apple, zzyxv) \in R\), but \((abba, ababb) \notin R\).

(a) Briefly describe an algorithm that takes as input two such strings, \(s\) and \(t\), and returns 1 if \((s, t) \in R\) or 0 if \((s, t) \notin R\).

Solution. We compare the strings \(s\) and \(t\) letter by letter from left to right, continuing if the corresponding letters are equal, until either the corresponding letters are not equal, or we reach the end of one or both strings. We have cases:

i. If the corresponding unequal letters are \(s_i\) and \(t_i\), then we output 1 if the letter \(s_i\) precedes the letter \(t_i\) in alphabetic order, and otherwise output 0.

ii. If we reach the end of both strings simultaneously, then \(s = t\) and we output 1 (for reflexivity.)

iii. If we reach the end of \(s\) before the end of \(t\), then we output 1, and otherwise output 0.

(b) Prove that \(R\) is a total order, using the characterization you gave in part (a).

We first need to show that \(R\) is a partial order. If \(s\) is any string, then the algorithm outputs 1 for input \((s, s)\), so \((s, s) \in R\), and \(R\) is reflexive. If \(s\) and \(t\) are any strings such that both \((s, t) \in R\) and \((t, s) \in R\), then in the letter by letter comparison, neither string can end before the other, and there cannot be a pair of unequal letters, because otherwise there would be a 1 output and a 0 output. Thus, we must have \(s = t\), and \(R\) is antisymmetric. If \(s\), \(t\), and \(u\) are any strings such that \((s, t) \in R\) and \((t, u) \in R\) then clearly, if \(s = t\) or \(t = u\), we have \((s, u) \in R\). If \(s \neq t\) and \(t \neq u\), consider a left to right letter by letter comparison of all three strings, continuing as long as all three corresponding letters are equal.
The only ways the comparisons can end is if $s$ ends before $t$ and $u$ do, in which case $(s, u) \in R$, or the triple of letters is $(x, y, y)$ or $(x, x, y)$ where $x$ precedes $y$, in which case $(s, u) \in R$, or the triple of letters is $(x, y, z)$ where $x$ precedes $y$ and $y$ precedes $z$, in which case $(s, u) \in R$. Thus, $R$ is transitive, so $R$ is a partial order.

To see that $R$ is a total order, we consider any pair $s$ and $t$ of nonempty finite strings. We must show that they are comparable, that is, $(s, t) \in R$ or $(t, s) \in R$. Clearly, if $s = t$, then the algorithm outputs 1 and we have $(s, t) \in R$. If $s \neq t$, then when we compare them letter by letter either we reach the end of one of them (and not the other) or there is some first corresponding pair of letters, $s_i$ in $s$ and $t_i$ in $t$, that are different. If we reach the end of $s$ first, then the algorithm outputs 1 for $(s, t)$ and $(s, t) \in R$, while if we reach the end of $t$ first, the algorithm outputs 1 for $(t, s)$ and $(t, s) \in R$. If we find a corresponding pair $s_i$ and $t_i$ of letters that are different, then the algorithm outputs 1 for $(s, t)$ if $s_i$ precedes $t_i$ in alphabetic order, so $(s, t) \in R$, and otherwise the algorithm outputs 1 for $(t, s)$, so $(t, s) \in R$. In any case, we have shown $(s, t) \in R$ or $(t, s) \in R$, so $s$ and $t$ are comparable, and $R$ is a total order.

(c) Prove or disprove: every nonempty set $S \subseteq A$ has a minimum element with respect to the total order $R$.

Solution. This is false. As a counterexample we consider the infinite set $S$ containing all strings that start with $n$ a’s and end with a b, for all positive integers $n$. That is,

$$S = \{ab, aab, aaab, aaaaab, \ldots\}.$$

This set is clearly nonempty, but has no minimum element. For any element of $S$, for example, $a^n b$ (that is, $n$ a’s followed by a b), there exists another element of $S$, for example $a^{n+1} b$ (that is, $(n + 1)$ a’s followed by a b) that precedes it in the ordering $R$.

5. (15 points) Let $A$ be any set and $R$ a relation on $A$ that is a partial order. Prove that for all integers $n \geq 2$, there do not exist $n$ distinct elements $a_1, a_2, \ldots, a_n \in A$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2, \ldots, n-1$ and $(a_n, a_1) \in R$.

The proof is by induction on $n$. The predicate $P(n)$ is that there do not exist $n$ distinct elements $a_1, a_2, \ldots, a_n$ of $A$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2, \ldots, n-1$ and $(a_n, a_1) \in R$.

The base case is $n = 2$. The negation of $P(2)$ is that there exist $a_1$ and $a_2$ in $A$ such that $a_1 \neq a_2$, $(a_1, a_2) \in R$ and $(a_2, a_1) \in R$. This would be a violation of antisymmetry for the partial order $R$, so $P(2)$ must be true.

Assume $P(n)$ is true for some $n \geq 2$. We show that assuming $P(n + 1)$ is false leads to a contradiction. If $P(n + 1)$ is false, then there exist $(n + 1)$ distinct elements $a_1, a_2, \ldots, a_n, a_{n+1}$ of $A$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2, \ldots, n$, and $(a_{n+1}, a_1) \in R$. Thus, $(a_n, a_{n+1}) \in R$ and $(a_{n+1}, a_1) \in R$. Because $R$ is a partial order, it is transitive, and therefore $(a_n, a_1) \in R$. Thus, there exist $n$ distinct elements $a_1, a_2, \ldots, a_n$ of $A$ such that $(a_i, a_{i+1}) \in R$ for $i = 1, 2, \ldots, n-1$ and $(a_n, a_1) \in R$, contradicting $P(n)$. Thus, $P(n + 1)$ must be true, so by mathematical induction we have proved $(\forall n \geq 2)P(n)$.

6. (20 points) Let $A$ be the set of all functions $f : \mathbb{N} \to \mathbb{N}$. Note that if $f, g \in A$, then $f = g$ if and only if $f(n) = g(n)$ for all $n \in \mathbb{N}$.

(a) Let $R_1$ be the set of all pairs $(f, g) \in A \times A$ such that $f(n)$ is in $O(g(n))$. Prove that $R_1$ is reflexive and transitive, but not antisymmetric, and thus is not a partial order.
Let $R_f, g$ transitive, suppose $(f, g) \in R_1$ and $(g, h) \in R_1$. Then $f(n)$ is in $O(g(n))$ and $g(n)$ is in $O(h(n))$. Thus there exist constants $c_1 > 0$, $N_1$, $c_2 > 0$, and $N_2$ such that for all $n > N_1$, $|f(n)| \leq c_1 \cdot |g(n)|$,

and for all $n > N_2$, $|g(n)| \leq c_2 \cdot |h(n)|$,

which implies that for all $n > \max(N_1, N_2)$, $|f(n)| \leq c_1 \cdot c_2 |h(n)|$.

Thus, choosing $c = c_1 \cdot c_2 > 0$ and $N = \max(N_1, N_2)$, we have $f(n)$ is in $O(h(n))$, so $(f, h) \in R_1$, and $R_1$ is transitive.

To see that $R_1$ is not antisymmetric, note that the functions $f(n) = n$ and $g(n) = 2n$ are in $A$, and $f(n)$ is in $O(g(n))$ (using $c = 1$ and $N = 0$) and $g(n)$ is in $O(f(n))$ (using $c = 2$ and $N = 0$), so $(f, g) \in R_1$ and $(g, f) \in R_1$. However, $f \neq g$ because $f(1) = 1$ while $g(1) = 2$.

(b) Let $R_2$ be the set of all pairs $(f, g) \in A \times A$ such that $f(n)$ is in $\Theta(g(n))$. Prove that $R_2$ is an equivalence relation.

Solution. We know that $f(n)$ is in $\Theta(g(n))$ if and only if $f(n)$ is in $O(g(n))$ and $g(n)$ is in $O(f(n))$. Thus $(f, g) \in R_2$ if and only if $(f, g) \in R_1$ and $(g, f) \in R_1$. In part (a) we showed $R_1$ is reflexive. Thus for any $f \in A$, $(f, f) \in R_1$, so $(f, f) \in R_2$, and $R_2$ is reflexive. Clearly, $(f, g) \in R_2$ if and only if $(f, g) \in R_1$ and $(g, f) \in R_1$, which is true if and only if $(g, f) \in R_2$, so $R_2$ is symmetric. To see that $R_2$ is transitive, assume that $(f, g) \in R_2$ and $(g, h) \in R_2$. Then $(f, g) \in R_1$ and $(g, f) \in R_1$, as well as $(g, h) \in R_1$ and $(h, g) \in R_1$. In part (a) we showed that $R_1$ is transitive, so this implies $(f, h) \in R_1$ and $(h, f) \in R_1$, so $(f, h) \in R_2$, and $R_2$ is transitive. Because $R_2$ is reflexive, symmetric and transitive, $R_2$ is an equivalence relation.