Please submit your homework in two separate parts: problems 1-3 in the first part, and 4-6 in the second part. Make sure your name is on both parts. In the first part, please include the names of anyone (including course staff) you consulted with in connection with this assignment, as well as listing any resources (including course materials) you consulted.

Graphs and trees in these problems are assumed to have at least one vertex. The degree sequence of a simple undirected graph is the sequence of degrees of its vertices, sorted into non-increasing order. For example, if $G = (V,E)$ is the simple undirected graph with vertices $V = \{1, 2, 3, 4, 5, 6\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{4, 5\}\}$ then the degree sequence of $G$ is $(3, 2, 2, 2, 1, 0)$.

To construct proofs requested below, you may use definitions and results about graphs and trees given in Chapter 10 of “Notes on Discrete Mathematics” and things you prove yourself, but not facts about graphs and trees from elsewhere. Please cite by number the Lemmas and Theorems you use from Chapter 10.

**PART I**

1. (15 points) For each of the following sequences, determine whether it is the degree sequence of some simple undirected graph with 6 vertices. If it is, draw one such graph; if it isn’t, prove that it is not. (Hint: what does a degree sequence tell you about the number of edges in the graph?)

(a) $(2,2,2,1,1,1)$

*Solution.* There is no graph with this degree sequence. If there were such a graph, the sum of its degrees would be 9, an odd number, in violation of the Handshaking Lemma (10.9.3), which implies that the sum must be an even number.

(b) $(2,2,1,1,1,1)$

*Solution.* One such graph is:

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(c) (5,3,3,2,2,1)

Solution. One such graph is:

(d) (3,3,3,3,2,2)

Solution. One such graph is:

(e) (5,4,3,3,1,0)

Solution. There is no such graph. If there were, it would have a vertex of degree 5, which must be adjacent to every other vertex, and a vertex of degree 0, which must not be adjacent to any other vertex, which is a contradiction.

2. (15 points) For each of the following sequences, determine (i) whether or not it is the degree sequence of some tree, AND (ii) whether or not it is the degree sequence of some non-tree. If there is a tree, draw one such; if not, prove there is no such tree. And if there is a non-tree, draw one such; if not, prove there is no such non-tree. There may be one or the other, or both, or neither.

(a) (3,3,2,2,1,1)

Solution.

(i) This is not the degree sequence of any tree. If $G$ is a graph with this degree sequence, then the sum of its degrees is 12, which by the Handshaking Lemma (10.9.3) implies that $G$ has 6 edges. However, $G$ has 6 vertices, and we know that the number of edges in any tree is one less than the number of vertices (Theorem 10.9.8), so $G$ cannot be a tree.

(ii) One such non-tree is:

(b) (3,3,2,2,1,1,1,1)

Solution.

(i) One such tree is:

(ii) One such non-tree is:

(c) (4,3,2,1,1,1,1,1)

Solution.

(i) One such tree is:
3. (20 points) Prove the following statements.

(a) Let \( G = (V, E) \) be a tree with at least one edge \( uv \). Let \( G_1 = (V, E_1) \) where \( E_1 = E \setminus \{uv\} \). Then \( G_1 \) contains exactly two connected components.

Solution. Because \( G \) is a tree, for any vertices \( s, t \in V \), there is a unique simple path from \( s \) to \( t \) in \( G \), by Lemma 10.9.4. Define the binary relation \( R \) on \( V \) to contain all pairs of vertices \((s, t)\) such that the unique simple path from \( s \) to \( t \) in \( G \) does not use the edge \( uv \).

Then \((s, t) \in R\) iff \( s \) is connected to \( t \) in \( G_1 \). To see this, note that if \((s, t) \in R\), there is a path in \( G \) from \( s \) to \( t \) that does not use the edge \( uv \), which will also be a path in \( G_1 \) from \( s \) to \( t \), so \( s \) is connected to \( t \) in \( G_1 \). Conversely, if \( s \) is connected to \( t \) in \( G_1 \), there is a path (which we may take to be simple, by Lemma 9.5.1) of edges from \( E_1 \) from \( s \) to \( t \), which is a simple path in \( G \) from \( s \) to \( t \) that does not use edge \( uv \), so \((s, t) \in R\).

Thus \( R \) is the equivalence relation of connectivity in \( G_1 \), and its equivalence classes are the connected components of \( G_1 \). Note that \((u, v) \notin R\), because the unique simple path from \( u \) to \( v \) in \( G \) is the edge \( uv \). Thus, \( R \) has at least 2 equivalence classes.

We claim that every \( x \in V \) is in the equivalence class of \( u \) or in the equivalence class of \( v \), which will show that \( R \) has exactly 2 equivalence classes. Suppose \( x \) is not in the equivalence class of \( v \), that is, the unique simple path from \( x \) to \( v \) in \( G \) uses the edge \( uv \). Then \( u \) occurs before \( v \) on the path (\( u \) and \( v \) each occur just once because the path is simple), so there is a simple path from \( x \) to \( u \) that does not use the edge \( uv \), so \((x, u) \in R\), and \( x \) is in the equivalence class of \( u \).

(b) Let \( G = (V, E) \) be a tree with two distinct vertices \( u \) and \( v \) such that \( uv \notin E \). Let \( G_2 = (V, E_2) \) where \( E_2 = E \cup \{uv\} \). Then \( G_2 \) contains a simple cycle.

Solution. Since \( G \) is a tree, there is a unique simple path of length \( k \), say

\[ u = v_0, v_1, \ldots, v_k = v \]

from \( u \) to \( v \) in \( G \), by Lemma 10.9.4. Because \( u \) and \( v \) are distinct vertices and \( uv \) is not an edge of \( G \), we must have \( k \geq 2 \). When we add the edge \( uv \) to \( G \) to get \( G_2 \), we find that

\[ v_0, v_1, \ldots, v_k \]
is a simple cycle in \( G_2 \), because \( k \geq 2 \), all the vertices are distinct, and \( v_0 = u \) and \( v_k = v \), so there is an edge from \( v_k \) to \( v_0 \) in \( G_2 \).

PART II

4. (15 points) Describe a recursive algorithm that takes an integer \( n \geq 2 \) as input and outputs a Hamiltonian cycle in the cube \( Q_n \) (see Section 10.4 in the text.) Give an inductive proof of the correctness of your algorithm.

Solution.

The algorithm QHC takes an integer \( n \geq 2 \) as input. If \( n = 2 \), it outputs the sequence \( 00, 01, 11, 10 \), which is a Hamiltonian cycle in the cube \( Q_2 \). If \( n > 2 \), it calls itself recursively with \( (n - 1) \) to get a sequence of \( N = 2^{n-1} \) strings of \( (n - 1) \) bits each, say \( s_1, s_2, \ldots, s_N \).

Then it outputs the sequence of \( 2N = 2^n \) strings of \( n \) bits each, say \( t_1, t_2, \ldots, t_{2N} \) defined by \( t_i = 0s_i \) if \( 1 \leq i \leq N \), and \( t_i = 1s_{2N-i+1} \) if \( N + 1 \leq i \leq 2N \). That is, the first \( N \) elements are obtained by prefixing the elements of the sequence for \( (n - 1) \) by 0 and the second \( N \) elements are obtained by prefixing the elements of the reverse of the sequence for \( (n - 1) \) by 1.

For example, for \( n = 3 \) we get the sequence:

\[
000, 001, 011, 010, 110, 111, 101, 100.
\]

We prove by induction on \( n \) that QHC on input \( n \) outputs a Hamiltonian cycle for \( Q_n \), for all integers \( n \geq 2 \). The base case is \( n = 2 \), for which the output is a Hamiltonian cycle for \( Q_2 \) by inspection. The inductive hypothesis is that QHC on some input \( (n - 1) \geq 2 \) outputs a Hamiltonian cycle for \( Q_{n-1} \). When QHC is called on input \( n \), we have \( n > 2 \) and it will call itself recursively on \( (n - 1) \), and by the inductive hypothesis, the result, \( s_1, s_2, \ldots, s_N \) is a Hamiltonian cycle for \( Q_{n-1} \), where \( N = 2^{n-1} \). That is, every vertex of \( Q_{n-1} \) appears exactly once in this list, and for every \( i \) such that \( 1 \leq i \leq N - 1 \), there is an edge from \( s_i \) to \( s_{i+1} \), and finally there is an edge from \( s_N \) to \( s_1 \).

By prefixing the elements of this list first by 0 and then by 1, we generate all \( 2N \) distinct vertices of \( Q_n \). To see that every required pair is joined by an edge, we have the cases:
(a) For $1 \leq i \leq N - 1$, the pair $s_i$ and $s_{i+1}$ differ in one position, so $t_i = 0$ and $t_{i+1} = 0$. The pair $t_N = 0s_N$ and $t_{N+1} = 1s_N$ differ in one position (the leftmost position.)

(b) For $i = N$, the pair $t_N = 0s_N$ and $t_{N+1} = 1s_N$ differ in one position (the leftmost position.)

(c) For $N + 1 \leq i \leq 2N - 1$, the pair $t_i = 1s_{2N-i+1}$ and $t_{i+1} = 1s_{2N-i}$ differ in one position, because $s_{2N-i}$ and $s_{2N-i+1}$ differ in one position.

(d) Finally, $t_{2N} = 1s_1$ differs from $t_1 = 0s_1$ in one position (the leftmost position.)

Thus, by induction we conclude that QHC with input $n$ outputs a Hamiltonian cycle for the cube $Q_n$ for every integer $n \geq 2$.

5. (15 points) If $G = (V, E)$ is a simple undirected graph, then its complement is the simple undirected graph $G' = (V, E')$ where $E'$ is the set of all $uv$ such that $u$ and $v$ are distinct vertices in $V$ and $uv \notin E$. Prove or disprove: For every tree $G$ with at least two vertices, its complement $G'$ is connected if and only if $G$ does not contain any vertex of degree $|V| - 1$.

Solution.
This is true. Let $G = (V, E)$ be any tree with $|V| \geq 2$.

If $G$ contains a vertex $u$ of degree $|V| - 1$, then $u$ is adjacent to every other vertex of $G$. In the complement graph $G'$, $u$ is not adjacent to any other vertex, and since $|V| \geq 2$, this means that $G'$ is not connected.

If $G'$ is not connected, then it contains at least two connected components, say $C_1$ and $C_2$. Then $C_1$ or $C_2$ must consist of just one vertex. To see this, suppose $u_1$ and $v_1$ are distinct vertices in $C_1$ and $u_2$ and $v_2$ are distinct vertices in $C_2$. Thus, $u_1u_2$, $u_2v_1$, $v_1v_2$, $v_2u_1$ are not edges in $G'$ (otherwise, $C_1$ and $C_2$ would not be separate connected components of $G'$), so they must be edges in $G$, which is a contradiction because they form a cycle in $G$, which is a tree and by definition acyclic. Suppose $C_1$ has just one vertex $u$. Then, for every other $v \in V$, $uv$ is not an edge of $G'$, and is an edge of $G$. Thus, $u$ is a vertex of degree $|V| - 1$ in $G$.

6. (20 points) A $k$-coloring of a simple undirected graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots, k\}$ with the property that for all vertices $u$ and $v$ in $V$, if $uv \in E$, then $c(u) \neq c(v)$. (That is, any two adjacent vertices are assigned different colors.)

(a) Prove that a simple undirected graph $G$ can be colored with $k = 2$ colors if and only if it contains no cycle of odd length.

Solution.
Let $G = (V, E)$ be a simple undirected graph. Assume $G$ contains a cycle of odd length, say $v_1, v_2, \ldots, v_k$. 
where \( k \geq 3 \) is an odd integer. Suppose for the sake of contradiction that \( c : V \to \{1, 2\} \) is a correct coloring of \( G \). Without loss of generality, we may assume \( c(v_1) = 1 \). (Otherwise, we could reverse the color of every vertex.) Then we must have \( c(v_2) = 2 \) and \( c(v_3) = 1 \), and so on, that is, \( c(v_i) = 1 \) iff \( i \) is odd. But both \( k \) and 1 are odd, so the adjacent vertices \( v_k \) and \( v_1 \) are both colored with 1, a contradiction.

Conversely, assume that \( G \) contains no odd cycles. We give a procedure to construct a coloring of \( G \) with 2 colors. It suffices to do so when \( G \) is connected, because the colorings of the connected components of \( G \) can be constructed independently.

When \( G \) is connected, the procedure selects an arbitrary start vertex \( s \) of \( G \) and assigns it color 1. If there is any edge \( uv \) such that \( u \) has been assigned a color and \( v \) has not, then \( v \) is assigned the opposite of \( u \)'s color (that is, \( v \) is assigned 2 if \( u \) is assigned 1 and vice versa.) If a vertex \( v \) is assigned the color 2, then there is an odd-length path from \( s \) to \( v \), and if \( v \) is assigned the color 1, then there is an even-length path from \( s \) to \( v \).

This procedure halts only when all vertices have been assigned a color. To see this, if \( v \) has not been assigned a color, then there is a path from \( s \) to \( v \) and for some edge \( uv \) along this path, \( u \) has been assigned a color and \( v \) has not.

To see that this is a correct coloring, assume for the sake of contradiction that there is an edge \( uv \) such that \( u \) and \( v \) are assigned the same color. If the color is 2, then there are paths of odd length from \( s \) to \( u \) and \( s \) to \( v \), which, together with the edge \( uv \) gives an odd-length closed walk from \( s \) to \( u \) to \( v \) and back to \( s \). Similarly, if the color is 1 then there are paths of even length from \( s \) to \( u \) and \( s \) to \( v \), which, together with the edge \( uv \) gives an odd-length closed walk from \( s \) to \( u \) to \( v \) and back to \( s \).

We conclude by showing that an odd-length closed walk in \( G \) implies an odd-length simple cycle in \( G \). This is proved analogously to Lemma 10.9.2. The base case is a closed walk of length 3, which must be a simple cycle. For \( n > 3 \), if the walk is a simple cycle, we are done. Otherwise, the walk is

\[
v_1, \ldots, v_k
\]

and for some \( i < j \) we have \( v_i = v_j \). Then one of the two closed walks

\[
v_1, \ldots, v_i, v_{j+1}, \ldots, v_k,
\]

which removes the segment from \( v_{i+1} \) to \( v_j \), or

\[
v_{i+1}, \ldots, v_j,
\]

which keeps the segment from \( v_{i+1} \) to \( v_j \), is a shorter odd-length closed walk, which concludes the inductive proof.
(b) Let $G = (V, E)$ be a simple undirected graph. The directed graph $H = (V, F)$ is an acyclic orientation of $G$ if $H$ is acyclic and $F$ contains exactly one of the directed edges $uv$ or $vu$ for every undirected edge $uv \in E$, and $F$ contains no other edges. Show that if $H$ is an acyclic orientation of $G$ in which every directed path in $H$ has length at most $k$, then $G$ can be colored with $k + 1$ colors.

Solution. Assume that $H$ is an acyclic orientation of $G$ such that every directed path in $H$ has length at most $k$. For any vertex $v \in V$, define $\ell(v)$ to be the length of the longest directed path from any vertex $u$ to $v$ in $H$. Then $\ell$ is a function that maps $V$ to $\{0, 1, \ldots, k\}$, because the longest path in $H$ has length at most $k$. For any vertex $v \in V$, define $c(v) = \ell(v) + 1$. Then $c$ is a function that maps $V$ to $\{1, 2, \ldots, k + 1\}$, and we claim it is a coloring of $G$.

To see this, we must show that if $uv \in E$, then $c(u) \neq c(v)$. Because $H$ is an acyclic orientation of $G$ then either $uv$ or $vu$ (but not both) is a directed edge of $H$. Without loss of generality, suppose that $uv$ is a directed edge of $H$. Then the longest path in $H$ to $v$ is at least one longer than the longest path in $H$ to $u$, because a path to $u$ could be extended to $v$ by adding the edge $uv$. Thus, $\ell(u) < \ell(v)$, so $c(u) < c(v)$, and $c(u) \neq c(v)$. Hence $c$ is a coloring of $G$ with $k + 1$ colors.