Computer Science 202
Homework #8, due in class Thursday, Nov. 17, 2016

Please submit your homework in two separate parts: problems 1-3 in the first part, and 4-6 in the second part. Make sure your name is on both parts. In the first part, please include the names of anyone (including course staff) you consulted with in connection with this assignment, as well as listing any resources (including course materials) you consulted.

Graphs and trees in these problems are assumed to have at least one vertex. The degree sequence of a simple undirected graph is the sequence of degrees of its vertices, sorted into non-increasing order. For example, if $G = (V, E)$ is the simple undirected graph with vertices

$$V = \{1, 2, 3, 4, 5, 6\}$$

and edges

$$E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{4, 5\}\}$$

then the degree sequence of $G$ is

$$(3, 2, 2, 2, 1, 0).$$

To construct proofs requested below, you may use definitions and results about graphs and trees given in Chapter 10 of “Notes on Discrete Mathematics” and things you prove yourself, but not facts about graphs and trees from elsewhere. Please cite by number the Lemmas and Theorems you use from Chapter 10.

PART I

1. (15 points) For each of the following sequences, determine whether it is the degree sequence of some simple undirected graph with 6 vertices. If it is, draw one such graph; if it isn’t, prove that it is not. (Hint: what does a degree sequence tell you about the number of edges in the graph?)

   (a) (2,2,1,1,1)
   (b) (2,2,1,1,1)
   (c) (5,3,3,2,2,1)
   (d) (3,3,3,3,2,2)
   (e) (5,4,3,3,1,0)

2. (15 points) For each of the following sequences, determine (i) whether or not it is the degree sequence of some tree, AND (ii) whether or not it is the degree sequence of some non-tree. If there is a tree, draw one such; if not, prove there is no such tree. And if there is a non-tree, draw one such; if not, prove there is no such non-tree. There may be one or the other, or both, or neither.
3. (20 points) Prove the following statements.

(a) Let $G = (V, E)$ be a tree with at least one edge $uv$. Let $G_1 = (V, E_1)$ where $E_1 = E \setminus \{uv\}$. Then $G_1$ contains exactly two connected components.

(b) Let $G = (V, E)$ be a tree with two distinct vertices $u$ and $v$ such that $uv \notin E$. Let $G_2 = (V, E_2)$ where $E_2 = E \cup \{uv\}$. Then $G_2$ contains a simple cycle.

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**PART II**

4. (15 points) Describe a recursive algorithm that takes an integer $n \geq 2$ as input and outputs a Hamiltonian cycle in the cube $Q_n$ (see Section 10.4 in the text.) Give an inductive proof of the correctness of your algorithm.

5. (15 points) If $G = (V, E)$ is a simple undirected graph, then its complement is the simple undirected graph $G' = (V, E')$ where $E'$ is the set of all $uv$ such that $u$ and $v$ are distinct vertices in $V$ and $uv \notin E$. Prove or disprove: For every tree $G$ with at least two vertices, its complement $G'$ is connected if and only if $G$ does not contain any vertex of degree $|V| - 1$.

6. (20 points) A $k$-coloring of a simple undirected graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots, k\}$ with the property that for all vertices $u$ and $v$ in $V$, if $uv \in E$, then $c(u) \neq c(v)$. (That is, any two adjacent vertices are assigned different colors.)

(a) Prove that a simple undirected graph $G$ can be colored with $k = 2$ colors if and only if it contains no cycle of odd length.

(b) Let $G = (V, E)$ be a simple undirected graph. The directed graph $H = (V, F)$ is an acyclic orientation of $G$ if $H$ is acyclic and $F$ contains exactly one of the directed edges $uv$ or $vu$ for every undirected edge $uv \in E$, and $F$ contains no other edges. Show that if $H$ is an acyclic orientation of $G$ in which every directed path in $H$ has length at most $k$, then $G$ can be colored with $k + 1$ colors.