YOUR NAME PLEASE:

**SOLUTIONS**

Computer Science 202
Midterm
1-2:15 pm, 18 October 2016

Closed book and closed notes. No electronic devices. Show ALL work you want graded ON THE TEST ITSELF.

For problems that do not ask you to justify the answer, an answer alone is sufficient. However, if the answer is wrong and no derivation or supporting reasoning is given, there will be no partial credit.

Recall that $\mathbb{N}$ denotes the set of natural numbers (integers greater than or equal to zero) and $\mathbb{Z}$ denotes the set of all integers (positive, negative and zero).

GOOD LUCK!
1. (10 points) Give a succinct definition of each of the following concepts. Assume $A$, $B$ and $C$ are sets of positive integers, $f$ is a function with domain $A$ and co-domain $B$, and $g$ is a function with domain $B$ and co-domain $C$.

(a) The union of $A$ and $B$, denoted $A \cup B$.

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

(b) The function $f$ is injective.

For all $a_1, a_2 \in A$,

if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

(c) The set difference of $A$ and $B$, denoted $A \setminus B$.

$$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$$

(d) The power set of $A$, denoted $\mathcal{P}(A)$.

$$\mathcal{P}(A) = \{ S \mid S \subseteq A \}$$

(e) The composition of $g$ and $f$, denoted $(g \circ f)$.

$(g \circ f)$ is the function with domain $A$ and co-domain $C$ such that for all $a \in A$, $(g \circ f)(a) = g(f(a))$
2. (10 points) Determine whether the following statements are true for ALL sets of positive integers $A$, $B$, and $C$. Your answer should be a proof or a counterexample, NOT a Venn diagram.

(a) If $(A \setminus B) = (A \setminus C)$ and $(B \setminus A) = (C \setminus A)$, then $B \subseteq C$.

True.

Assume $(A \setminus B) = (A \setminus C)$ and $(B \setminus A) = (C \setminus A)$.

Let $b$ be an arbitrary element of $B$.

Cases:

1. $b \in A$.

Then $b \notin (A \setminus B)$ so $b \notin (A \setminus C)$ and thus $b \notin C$ because $b \in A$.

2. $b \notin A$.

Then $b \in (B \setminus A)$ so $b \in (C \setminus A)$, so $b \in C$.

In either case, $b \in C$. Since $b \in B$ was arbitrary, this shows $B \subseteq C$.

(b) If $\mathcal{P}(A) \subseteq \mathcal{P}(B) \cup \mathcal{P}(C)$ then $A \subseteq B \cap C$.

False.

Consider the sets

$A = B = \{1, 2\}$ and $C = \emptyset$.

Then

$\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset, \{1, 2\}\}$ and $\mathcal{P}(C) = \{\emptyset\}$.

Thus, $\mathcal{P}(A) \subseteq \mathcal{P}(B) \cup \mathcal{P}(C)$.

However, $B \cap C = \emptyset$ and $A \notin B \cap C$ because $1 \in A$ and $1 \notin B \cap C$.

Thus, this claim does not hold for all sets of three positive integers $A$, $B$, and $C$. 

3. (12 points) For each of the following functions from the positive integers to the positive integers, give a table of the values of the function for $1 \leq n \leq 6$. Then state whether the function is injective and whether it is surjective, and justify your answers.

(a) $f(n)$ is $n - 1$ if $n$ is even, or $2n$ if $n$ is odd.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>3</td>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

$f$ is injective. $f$ maps odd numbers to even numbers and even numbers to odd numbers. Thus if $f(m) = f(n)$ then both $m$ and $n$ are even or both are odd.

If both are even, $m - 1 = n - 1$, so $m = n$.

If both are odd, $2m = 2n$, so $m = n$.

In either case, $f(m) = f(n)$ implies $m = n$.

$f$ is not surjective. If $f(n) = 4$ then we must have $2n = 4$ and $n$ odd, a contradiction.

(b) $g(n)$ is the largest element of the set {$j \in \mathbb{Z} | 3^j - 1 \leq n$}.

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>$g(n)$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

$g$ is not injective, for example $g(1) = 1 = g(2)$.

$g$ is surjective. Given any positive integer $m$, let $n = 3^{m-1}$, a positive integer. Then $m$ is the largest value of $j$ such that $3^j - 1 \leq n$, so $g(n) = m$. 


(Problem 3, continued)

(c) \( h(n) = \lceil (3n - 1)/2 \rceil \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
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<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
<td>( h(n) )</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
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Claim: \( h(n+1) \geq h(n)+1 \) for all integers \( n \geq 1 \).

Because \( h(n+1) = \lceil \frac{3(n+1)-1}{2} \rceil = \lceil \frac{3n-1+3}{2} \rceil \geq h(n)+1 \).

Then for all integers \( m,n \geq 1 \), if \( m > n \) then \( h(m) > h(n) \), so \( h \) is \underline{injective}.

Because \( h(1) = 1 \) and \( h(n) \geq 3 \) for all \( n \geq 2 \),

\( h(n) \neq 2 \) for all positive integers \( n \),

and \( h \) is \underline{not surjective}.

(d) \( k(n) \) is the number of positive integers that (evenly) divide \( n \).

<table>
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<tr>
<th>( n )</th>
<th>1</th>
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<th>4</th>
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</tr>
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<tbody>
<tr>
<td>( k(n) )</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
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</table>

\( k \) is \underline{not injective}, for example \( k(2) = k(3) \).

\( k \) is \underline{surjective}. Given any positive integer \( m \), let \( n = 2^m - 1 \) a positive integer. The positive integers that divide \( n \) are \( 1, 2, 4, 8, \ldots, 2^{m-1} \), so \( k(n) = m \).
4. (12 points) We recursively define the function \( g(n) \) for all \( n \in \mathbb{N} \) as follows. We set \( g(0) = 1 \) and \( g(1) = 3 \), and for all \( n \geq 2 \),
\[
g(n) = g(n-1) + g(n-2).
\]

Prove the following statement by strong mathematical induction, identifying the predicate \( P(n) \), the base case(s), and the inductive hypothesis.
\[
\forall n \in \mathbb{N} \left( g(n) \geq (3/2)^n \right)
\]

- **Base cases are**
  - \( P(0) \) is true because \( g(0) = 1 \geq \left( \frac{3}{2} \right)^0 \)
  - \( P(1) \) is true because \( g(1) = 3 \geq \frac{3}{2} = \left( \frac{3}{2} \right)^1 \)

- **Inductive hypothesis is**
  - Assume \( P(0), P(1), \ldots, P(n-1) \) are true, for some \( n \geq 2 \).

Then \( g(n) = g(n-1) + g(n-2) \), by definition of \( g \).

By the IH,
\[
g(n-1) \geq \left( \frac{3}{2} \right)^{n-1} \quad \text{and} \quad g(n-2) \geq \left( \frac{3}{2} \right)^{n-2}
\]

Thus \( g(n) \geq \left( \frac{3}{2} \right)^{n-1} + \left( \frac{3}{2} \right)^{n-2} \)

Factoring out \( \left( \frac{3}{2} \right)^{n-1} \) we get
\[
g(n) \geq \left( \frac{3}{2} \right)^{n-1} \left( 1 + \frac{3}{2} \right)
\]

Because \( 1 + \frac{3}{2} > \left( \frac{3}{2} \right) \), we have
\[
g(n) \geq \left( \frac{3}{2} \right)^{n-1} \cdot \left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^n
\]

So \( P(n) \) is true. Thus, for all \( n \geq 2 \),
\[
P(0) \land P(1) \land \cdots \land P(n-1) \rightarrow P(n) \quad \text{and} \quad \forall n \in \mathbb{N} \ P(n)
\]
5. We consider a domain that consists of all the animals in a certain zoo. We define the following predicates.

\[ E(x) \text{ means } x \text{ is an elephant} \]
\[ G(x) \text{ means } x \text{ is a giraffe} \]
\[ M(x) \text{ means } x \text{ is a monkey} \]
\[ H(x) \text{ means } x \text{ is happy} \]
\[ N(x) \text{ means } x \text{ is noisy} \]
\[ (x = y) \text{ means } x \text{ is equal to } y \]
\[ R(x, y) \text{ means } x \text{ respects } y \]

(a) With these predicates write the following statements as logical formulas.

i. (2 points) At least one monkey is happy and noisy.

\[ \exists x \left( H(x) \land H(x) \land N(x) \right) \]

ii. (2 points) Every giraffe respects every elephant.

\[ \forall x \left( G(x) \rightarrow \forall y \left( E(y) \rightarrow R(x,y) \right) \right) \]

\[ \text{or} \quad \forall x \forall y \left( G(x) \land E(y) \rightarrow R(x,y) \right) \]

iii. (2 points) There is an elephant that respects itself and respects no other animal.

\[ \exists x \left( E(x) \land \forall y \left( (x = y) \leftrightarrow R(x,y) \right) \right) \]

\[ \text{or} \quad \exists x \left( E(x) \land R(x,x) \land \forall y \left( R(x,y) \rightarrow (x = y) \right) \right) \]
(Problem 5, continued)

iv. (2 points) Every happy animal respects exactly those animals that respect all happy animals.

\[ \forall x \left( H(x) \rightarrow \forall y \left( R(x,y) \leftrightarrow \forall z \left( H(z) \rightarrow R(y,z) \right) \right) \right) \]

v. (2 points) Some animal respects no elephants.

\[ \exists x \forall y \left( E(y) \rightarrow \neg R(x,y) \right) \]

( or \[ \exists x \forall y \left( E(y) \land R(x,y) \right) \])

(b) (4 points) Does the last statement (v) in part (a) follow logically from the other statements (i-iv)? Please justify your answer by giving an informal proof or a counterexample (that is, an example of a zoo making (i-iv) true and (v) false.)

No, it doesn't follow logically.

Consider the following Zoo.

\[ \begin{array}{c|cccc} m & E & G & H & N \\
--- & --- & --- & --- & --- \\
m & 1 & 0 & 0 & 1 \\
e_1 & 0 & 1 & 0 & 0 \\
e_2 & 0 & 1 & 0 & 0 \\
g & 0 & 0 & 1 & 0 \\
\end{array} \]

\[ \begin{array}{c|cccc} \text{R(y,x)} & m & e_1 & e_2 & g \\
--- & --- & --- & --- & --- \\
m & 1 & 1 & 1 & 1 \\
e_1 & 0 & 1 & 0 & 0 \\
e_2 & 1 & 1 & 1 & 1 \\
g & 1 & 1 & 1 & 1 \\
\end{array} \]

(i) \(m\) is a happy, noisy monkey

(ii) \(g\) is the only giraffe, and it respects every elephant

(iii) \(e_1\) is an elephant that respects only itself

(iv) \(m\) is the only happy animal, and \(m, e_2, g\) are the animals that respect it, and it respects exactly those animals.

\(v\) is false because every animal respects at least one elephant.
6. (a) (4 points) Let \( f(n) \) and \( g(n) \) be functions with domain and co-domain \( \mathbb{N} \). Give the definition of the concept: \( f(n) \) is in \( \Omega(g(n)) \) iff there exist \( C > 0 \) and \( N \) such that for all \( n > N \),
\[
C \left| g(n) \right| \leq \left| f(n) \right|
\]

(b) (8 points) We define the function \( f(n) \) with domain and co-domain \( \mathbb{N} \) as follows.
For every \( n \in \mathbb{N} \), if \( n \) is even then \( f(n) = n/2 \) and if \( n \) is odd, then \( f(n) = 3n^2 + 1 \).
Prove that \( f(n) \) is in \( O(n^2) \), and that \( f(n) \) is not in \( \Omega(n^2) \).

Clearly \( \left| f(n) \right| = f(n) \) and \( |n^2| = n^2 \) for all \( n \in \mathbb{N} \).

(i) \( f(n) \) is in \( O(n^2) \).
Choose \( C = 4 \) and \( N = 1 \). Assume \( n > N \).
(a) if \( n \) is even, \( f(n) = \frac{n}{2} \) and 
\[
\frac{n}{2} \leq 4 \cdot n^2 \quad \text{because} \quad n > 1 \quad \text{and} \quad 1 \leq 8n.
\]
(b) if \( n \) is odd, \( f(n) = 3n^2 + 1 \) and 
\[
3n^2 + 1 \leq 4 \cdot n^2 \quad \text{because} \quad n > 1 \quad \text{and} \quad 1 \leq n^2.
\]

(ii) \( f(n) \) is not in \( \Omega(n^2) \).
Let \( C > 0 \) and \( N \) be given.
Choose \( n \) to be an even integer such that 
\[
N > N \quad \text{and} \quad n > \frac{1}{2C}.
\]
Then \( f(n) = \frac{n}{2} \) and \( \frac{n}{2} < C \cdot n^2 \) because
\[
\frac{1}{2C} < n. \quad \text{Thus, for all } C > 0 \quad \text{and } N,
\]
there exists \( n > N \) such that
\[
C \cdot |n^2| > |f(n)|, \quad \text{so } f(n) \text{ is not in } \Omega(n^2).
\]