Please turn in your homework in TWO SEPARATE PARTS: Part I is problems 1-3 and Part II is problems 4-6. Put your name on both parts. Please list (in Part I) any persons (including course staff) or resources you consulted with in connection with this assignment. Partial credit will be given if the grader can easily understand enough of your answer to award it.

PART I

1. (15 points) Construct truth tables for each of the following propositional expressions, and say for each whether it is a tautology, a contradiction or a contingency.

(a) \((p \land \neg q) \lor (\neg p \land q)\)
\[
\begin{array}{ccc}
 p & q & (p \land \neg q) \lor (\neg p \land q) \\
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{array}
\]
Contingency: true in some situations, false in others.

(b) \((p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)\)
\[
\begin{array}{ccc}
 p & q & (p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q) \\
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{array}
\]
Tautology: true in every situation.

(c) \((p \oplus q) \rightarrow ((p \oplus q) \oplus r)\)
\[
\begin{array}{ccc}
 p & q & r & (p \oplus q) \rightarrow ((p \oplus q) \oplus r) \\
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{array}
\]
Contingency: true in some situations, false in others.

(d) \((p \lor q) \rightarrow ((p \land r) \lor (q \land \neg r))\)
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<th>q</th>
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<th>(p ∨ q)</th>
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<th>((p ∧ r)</th>
<th>∨</th>
<th>(q ∧ ¬r))</th>
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Contingency: true in some situations, false in others.

(e) \((¬p ∧ q) ∧ (q ↔ p)\)

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<tr>
<th>p</th>
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<th>(¬p ∧ q)</th>
<th>∧</th>
<th>(q ↔ p)</th>
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Contradiction: false in every situation.

2. (15 points) Imagine that you have observed meetings involving five people, identified by the letters A, B, C, D, E, and you have induced the following rules about the meetings:

(a) If A and B are present then so is C or D (or both).
(b) If D is present then so are A and E.
(c) If C and E are present, then B is not present

Find all meetings of a group of four of these individuals that do not violate any of these rules. Justify your answer.

Answer. There are 5 groups of four of these individuals:

(a) ABCD violates rule (b)
(b) ABCE violates rule (c)
(c) ABDE satisfies all three rules
(d) ACDE satisfies all three rules
(e) BCDE violates rule (b) and (c)

Thus, ABDE and ACDE are the groups of four that do not violate any of the rules.

3. (20 points) Logician Raymond Smullyan describes the island of knights and knaves as populated by two kinds of people: those who always lie (knaves) and those who always tell the truth (knights). Imagine that you are visiting the island and you meet three of its inhabitants, conveniently named A, B and C.

For each of the following items, determine whether A, B, and C could make the statements attributed to them. If they could make the statements, determine all the assignments of types (knight or knave) to A, B, and C in which the list of statements is possible. Justify your answers.
(a) A says “B and C are knights.” B says “That is correct, C and I are knights.” C says “No, that is not correct!”

Answer. B and C contradict each other, and must be of different types. Both A and B claim that B and C are of the same type, which is not true, so A and B must be knaves. Since C is not the same type as B, C must be a knight. The assignment: A and B are knaves and C is a knight, is consistent with the statements, and is the only assignment that is.

(b) A says “There are an odd number of knights among us.” B says “Both A and I are knights.” C says “No, both A and B are knaves.”

Answer. B and C cannot both be knights. because their statements cannot both be true. If C is a knight, then A and B must be knaves, but then there are an odd number of knights (one), and what A says is true, which would contradict A being a knave. Hence C must be a knave. If B is a knight, then there would be an even number of knights (two), which would contradict A being a knight. Hence B must be a knave. In order for C’s statement to be false, A must be a knight. The assignment: A is a knight, B and C are knaves, is consistent with the statements, and is the only assignment that is.

(c) A says “Exactly two of us are knights.” B says “Exactly two of us are knaves.” C says “Not all of us are of the same type.”

Answer. If C is a knave, then C’s statement must be false, and all three must be the same type, namely knaves. Then all the statements are false, and the assignment: all three are knaves, is consistent.

If C is a knight, then there could be one or two knaves. There cannot be two knaves, because that would make B’s statement true, and B would be a knight, and there would be two knights, a contradiction. Thus, there must be just one knave, which makes A a knight and B a knave. This assignment: A and C are knights and B is a knave, is also consistent. These two assignments are the only consistent ones.

(d) A says “If I am a knave, then so is B.” B says “A and C are of the same type.” C says “A and B are of opposite types.”

Answer. If A is a knave, then the only way for what A says to be false is if B is a knight. In that case, A and C must both be knaves, but this means that what C says is true, contradicting the assignment of C as a knave.

If A is a knight, then A’s statement is automatically true (because \( F \rightarrow T \) and \( F \rightarrow F \) both evaluate to \( T \).) Then if B is a knight, C must also be a knight, but this is a contradiction, because what C says is false. If B is a knave, then what C says is true, so C must be a knight, and then what B says is true, contradicting the assignment of B as a knave.

Thus, there is no assignment consistent with these statements.
PART II

4. (15 points) For the following pairs of formulas, determine whether they are logically equivalent. Justify your answer.

(a) \((p \lor \neg q), \neg (p \rightarrow q)\)

Answer. These are not logically equivalent, as witnessed by the following line in their truth tables.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>((p \lor \neg q))</th>
<th>(\neg (p \rightarrow q))</th>
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(b) \((p \rightarrow (p \land q)), p \rightarrow q\)

Answer. These are logically equivalent, as witnessed by the equality of the columns of their truth tables.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>((p \rightarrow (p \land q)))</th>
<th>(p \rightarrow q)</th>
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(c) \(((p \lor q) \land (\neg p \lor r)), (q \land r)\)

Answer. These are not logically equivalent, as witnessed by the following line in their truth tables.

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<th>p</th>
<th>q</th>
<th>r</th>
<th>(((p \lor q) \land (\neg p \lor r)))</th>
<th>(q \land r)</th>
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(d) \((p \leftrightarrow (q \oplus r)), (r \rightarrow (p \oplus q))\)

Answer. These are not logically equivalent, as witnessed by the following line in their truth tables.

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<th>p</th>
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<th>((p \leftrightarrow (q \oplus r)))</th>
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(e) \(((p \rightarrow r) \land (q \rightarrow r)), ((p \lor q) \rightarrow r)\)

Answer. These are logically equivalent, witnessed by the equality of the columns in their truth tables.

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<tr>
<th>p</th>
<th>q</th>
<th>r</th>
<th>(((p \rightarrow r) \land (q \rightarrow r)))</th>
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5. (15 points) Translate each of the following statements about the domain of nonnegative integers into logical formulas, using the following function and predicate symbols. The function symbol \(a(x, y)\) stands for \(x + y\) and the function symbol \(m(x, y)\) stands for \(x \cdot y\). The predicate symbol \(L(x, y)\) stands for \(x \leq y\) and the predicate symbol \(E(x, y)\) stands for \(x = y\).
(a) For all $x$ and $y$ if $x \leq y$ and $y \leq x$ then $x = y$.  
Answer.  
\[ \forall x \forall y (L(x, y) \land L(y, x) \rightarrow E(x, y)) \]

(b) There does not exist a number such that every number is less than or equal to it.  
Answer.  
\[ \neg \exists x \forall y (L(y, x)) \]

(c) If $x$ is less than or equal to $y$ then there is a number $z$ such that $x + z$ is equal to $y$.  
Answer.  
\[ \forall x \forall y (L(x, y) \rightarrow \exists z (E(a(x, z), y))) \]

(d) If $x$ is less than or equal to $y$ then for any $z$, $x + z$ is less than or equal to $y + z$.  
Answer.  
\[ \forall x \forall y (L(x, y) \rightarrow \forall z (L(a(x, z), a(y, z)))) \]

(e) The product of any two numbers is always greater than or equal to at least one of them.  
Answer.  
\[ \forall x \forall y (L(x, m(x, y)) \lor L(y, m(x, y))) \]

6. (20 points) Choose and define function and predicate symbols, and translate each of the following arguments into statements in predicate logic. Taking all but the last statement of each argument as the hypotheses, and the last statement as the conclusion, assess the validity of the argument. If it is valid, give an informal proof that the conclusion is a logical consequence of the hypotheses. If it is invalid, specify an example domain and definitions of the function and predicate symbols that make all the hypotheses true and the conclusion false.

(a) (The domain is the animals in a particular zoo. Thus in $\forall x$ and $\exists x$ the variable $x$ refers to an animal in this zoo.)

i. Every animal that flies has wings.  
Answer. Define unary predicate symbols $F(x)$ for “$x$ flies”, $W(x)$ for “$x$ has wings”, $B(x)$ for “$x$ is a bird”, $H(x)$ for “$x$ has feathers”, and $S(x)$ for “$x$ has six legs”. Translations follow:  
i. Every animal that flies has wings.  
\[ \forall x (F(x) \rightarrow W(x)) \]

ii. Every bird has feathers and wings.  
\[ \forall x (B(x) \rightarrow (H(x) \land W(x))) \]

iii. No bird has six legs.  
\[ \neg \exists x (B(x) \land S(x)) \]

iv. There is an animal that has six legs and flies.  
\[ \exists x (S(x) \land F(x)) \]
v. There is an animal that has wings but is not a bird.
\[ \exists x (W(x) \land \neg B(x)) \]

This is a valid argument. By (iv), we know there is an animal that has six legs and flies; call one such animal \( a \). By (iii), no bird has six legs, so \( a \) is not a bird. By (i), because \( a \) flies, it must have wings. Thus, \( a \) is an animal that has wings and is not a bird, so at least one such animal exists, which is (v).

(b) (The domain is some nonempty set \( S \) of numbers. Thus in \( \forall x \) and \( \exists x \) the variable \( x \) refers to an element of this set.)

i. For all elements \( x \) and \( y \), if \( x \) is good for \( y \) then \( y \) is not good for \( x \).
\[ \forall x \forall y (G(x, y) \to \neg G(y, x)) \]

ii. For all elements \( x \), \( y \) and \( z \), if \( x \) is good for \( y \) and \( y \) good for \( z \) then \( x \) is good for \( z \).
\[ \forall x \forall y \forall z (G(x, y) \land G(y, z) \to G(x, z)) \]

iii. For every element \( x \) there is an element \( y \) that is good for \( x \).
\[ \forall x \exists y (G(y, x)) \]

iv. For every pair of elements \( x \) and \( y \) such that \( x \) is good for \( y \), there exists an element \( z \) such that \( x \) is good for \( z \) and \( z \) is good for \( y \).
\[ \forall x \forall y (G(x, y) \rightarrow \exists z (G(x, z) \land G(z, y))) \]

v. For every element \( x \) there is an element \( y \) such that \( x \) is good for \( y \).
\[ \forall x \exists y (G(x, y)) \]

Answer. We need one binary predicate: \( G(x, y) \) for “\( x \) is good for \( y \)”.

The translations of the statements follow.

i. For all elements \( x \) and \( y \), if \( x \) is good for \( y \) then \( y \) is not good for \( x \).
\[ \forall x \forall y (G(x, y) \rightarrow \neg G(y, x)) \]

ii. For all elements \( x \), \( y \) and \( z \), if \( x \) is good for \( y \) and \( y \) good for \( z \) then \( x \) is good for \( z \).
\[ \forall x \forall y \forall z (G(x, y) \land G(y, z) \rightarrow G(x, z)) \]

iii. For every element \( x \) there is an element \( y \) that is good for \( x \).
\[ \forall x \exists y (G(y, x)) \]

iv. For every pair of elements \( x \) and \( y \) such that \( x \) is good for \( y \), there exists an element \( z \) such that \( x \) is good for \( z \) and \( z \) is good for \( y \).
\[ \forall x \forall y (G(x, y) \rightarrow \exists z (G(x, z) \land G(z, y))) \]

v. For every element \( x \) there is an element \( y \) such that \( x \) is good for \( y \).
\[ \forall x \exists y (G(x, y)) \]

This argument is invalid. Consider the set \( S \) of all real numbers greater than or equal to 0, and let \( G(x, y) \) be true if \( x > y \). Then (i) is true because for all \( x \) and \( y \) in \( S \), if \( x > y \), it is not the case that \( y > x \). Statement (ii) true because for all \( x \), \( y \) and \( z \) in \( S \), if \( x > y \) and \( y > z \) then \( x > z \). Statement (iii) is true because for any \( x \) in \( S \), the number \( y = x + 1 \) is also in \( S \), and \( y > x \) is true. Statement (iv) is true because for any \( x \) and \( y \) in \( S \), if \( x > y \), then the number \( z = (x + y)/2 \) is also in \( S \) and satisfies \( x > z \) and \( z > y \). But statement (v) is not true, because for \( x = 0 \), we have \( x \in S \), but there is no \( y \in S \) such that \( x > y \).