Please turn in your homework in six separate parts, one for each problem. Put your name, the homework number, and the problem number on each part. Please list with problem 1 any persons (including course staff) or resources (including online) you consulted with in connection with this assignment.

The power set of the set $A$ is $\mathcal{P}(A)$. The integers are

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots \}.$$  

The natural numbers, or nonnegative integers, are

$$\mathbb{N} = \{0, 1, 2, 3, \ldots \}.$$  

PART I

1. (15 points) We define four sets as follows.

$$A = \{ x \in \mathbb{Z} \mid (-2 \leq x) \land (x \leq 4) \}$$
$$B = \{ x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} (x = y^2) \}$$
$$C = \{ x \in \mathbb{Z} \mid \lceil x/2 \rceil \geq 1 \}$$
$$D = \{ x \in A \mid \exists y \in \mathbb{Z} (x = 3y + 1) \}$$

Give each of the following sets by explicitly listing its elements using set braces and commas.

(a) $A =$  
(b) $A \cap B =$  
(c) $A \setminus C =$  
(d) $\mathcal{P}(D) =$  
(e) $(A \cap B) \times D =$

2. (15 points) Use the axiom of extensionality, the definitions of subset and the set operations, and case arguments to prove the following statements. Do not include Venn diagrams in your answers. Assume $A$, $B$, and $C$ are subsets of $\mathbb{Z}$.

(a) For all sets $A$, $B$, and $C$,

$$(A \cup C) \cap B \subseteq (A \cap B) \cup C.$$  

(b) For all sets $A$ and $B$,

$$(A \cup B) \subseteq (A \setminus B) \cup (B \setminus A) \cup (A \cap B).$$  

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(c) For all sets $A$, $B$, and $C$, if $(A \setminus C) \subseteq (B \setminus C)$ and $(B \setminus A) \subseteq (C \setminus A)$ and $(C \setminus B) \subseteq (A \setminus B)$ then \((A \cup B \cup C) \subseteq (A \cap B) \cup (B \cap C) \cup (A \cap C)\).

3. (20 points) We let $B = \mathcal{P}(\mathbb{N})$.

(a) There exists an injective function $f$ with domain $\mathbb{N}$ and co-domain $B$. Give an example of such an $f$.

(b) Show that the function $f$ you gave in part (a) is not surjective.

(c) Let $g$ be an arbitrary function with domain $\mathbb{N}$ and co-domain $B$. Let the set $S$ be given by the following.

\[
S = \{n \in \mathbb{N} \mid n \notin g(n)\}.
\]

Prove that for all $n \in \mathbb{N}$, $g(n) \neq S$.

(d) Assuming that (c) is true, why do this show that $\mathcal{P}(\mathbb{N})$ is uncountable?

4. (15 points) Let $A = \{x \in \mathbb{N} \mid 3$ does not divide $x\}$ and $B = \{x \in \mathbb{N} \mid x$ is even\}. Give functions $q$, $r$ and $s$ with domain $A$ and co-domain $B$ having the following properties, and prove that they have the properties stated.

(a) $q : A \rightarrow B$ is surjective but not injective.

(b) $r : A \rightarrow B$ is injective but not surjective.

(c) $s : A \rightarrow B$ is bijective.

5. (15 points) Let $f$ and $g$ be functions with domain and co-domain $\mathbb{Z}$. Prove or disprove each of the following statements. (Recall that $f \circ g$ is the composition of $f$ and $g$.)

(a) If $f$ and $g$ are injective then $(f + g)$ is injective.

(b) If $f$ and $g$ are surjective then $(f \circ g)$ is surjective.

(c) If $f$ and $g$ are bijective then $f \circ (g^{-1}) = g \circ (f^{-1})$.

6. (20 points) Let

\[
P = \mathbb{N} \times \mathbb{N}
\]

and for each $d \in \mathbb{N}$,

\[
P_d = \{(m, n) \in P \mid m + n = d\}.
\]

(a) Give $P_4$ as an explicitly listed finite set of elements.

(b) How many elements does $P_d$ have as a function of $d$?

(c) Prove that there exists an injective function from $P$ to $\mathbb{N}$.

(d) Explain why the truth of (c) proves that $P$ is countable.