1. (15 points) Prove that for all positive integers $n$

$$\sum_{i=1}^{n} (4i + 5) = 2n^2 + 7n.$$ 

*Solution.* By the linearity of summation (Lemma 6.1.2),

$$\sum_{i=1}^{n} (4i + 5) = 4 \sum_{i=1}^{n} i + 5 \sum_{i=1}^{n} 1.$$ 

Using two of the three “standard sums” from Section 6.4.1,

$$\sum_{i=1}^{n} (4i + 5) = 4 \frac{n(n + 1)}{2} + 5n,$$

and concluding with some algebraic manipulation, we have

$$\sum_{i=1}^{n} (4i + 5) = 2n^2 + 2n + 5n = 2n^2 + 7n.$$ 

(This could also be proved directly using mathematical induction)

2. We define $f(n) = \sum_{i=1}^{n} (-1)^{i} \cdot i$ for all natural numbers $n$.

(a) (5 points) Make a table of the values of $f(n)$ for $n = 0, 1, 2, 3, 4$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>-2</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) (10 points) Prove that for all natural numbers $n$

$$f(n) = (-1)^n \lceil n/2 \rceil.$$ 

*Solution.* We prove this by simple induction on the natural numbers. The predicate $P(n)$ is $f(n) = (-1)^n \lceil n/2 \rceil$. The base case is $n = 0$, where we have

$$f(0) = 0 = (-1)^0 \lfloor 0/2 \rfloor,$$
so $P(0)$ is true. The induction hypothesis (IH) is that $P(k)$ is true, for some $k \geq 0$. By the definition of $f(n)$ and the properties of summation,

$$f(k + 1) = f(k) + (-1)^{k+1}(k + 1).$$

We consider two cases for $k + 1$, namely even and odd.

- Suppose $k + 1$ is even. Then $k+1 = 2m$ for some natural number $m$, and $k = 2m - 1$, an odd number. Note that $(-1)^k = -1$ and

$$\left\lfloor k/2 \right\rfloor = \left\lfloor m - 1/2 \right\rfloor = m,$$

so by the IH, because $P(k)$ is true,

$$f(k) = (-1)m = -m.$$

Thus, using the relationship between $f(k + 1)$ and $f(k)$, we have

$$f(k + 1) = -m + (-1)^{k+1}(k + 1) = -m + 2m = m.$$

However,

$$(-1)^{k+1}[k+1]/2 = 1 \cdot m = m,$$

so in this case we have

$$f(k + 1) = (-1)^{k+1}[k+1]/2,$$

that is, $P(k + 1)$ is true.

- Suppose $k + 1$ is odd. Then $k+1 = 2m+1$ for some natural number $m$ and $(-1)^k = 1$ and

$$\left\lfloor k/2 \right\rfloor = \left\lfloor m \right\rfloor = m,$$

so by the IH, because $P(k)$ is true,

$$f(k) = m.$$

Again using the relationship between $f(k + 1)$ and $f(k)$, we have

$$f(k + 1) = m + (-1)^{k+1}(k + 1) = m + (-1)(2m + 1) = -(m + 1).$$

However,

$$(-1)^{k+1}[k+1]/2 = (-1)[m + 1/2] = -(m + 1),$$

so in this case also we have

$$f(k + 1) = (-1)^{k+1}[k+1]/2,$$

that is, $P(k + 1)$ is true.

In either case we have shown $P(k)$ implies $P(k + 1)$, which concludes the induction.

3. (20 points) Prove or disprove the statement in each part below.
(a) If $a, b, c, d$ are real numbers such that $a \leq b$ and $c \leq d$ then $a + c \leq b + d$.

Solution. This is true. Assume $a \leq b$ and $c \leq d$. Applying translation invariance for $\leq$ (Axiom 4.2.4), we have both $a + c \leq b + c$ and $c + b \leq d + b$. Because $+$ is commutative, by transitivity of $\leq$ (Axiom 4.2.3) we have $a + c \leq b + d$.

(b) If $a, b, c, d$ are real numbers such that $a \leq b$ and $c \leq d$ then $ac \leq bd$.

Solution. This is false. A counterexample is provided by $a = c = -1$ and $b = d = 0$. Because $-1 \leq 0$, we have $a \leq b$ and $c \leq d$, but $ac = 1$ and $bd = 0$, and the assertion $1 \leq 0$ is false.

(c) If $a$ and $b$ are real numbers such that $0 \leq a \leq b$ then for all positive integers $n$ we have $0 \leq a^n \leq b^n$.

Solution. This is true. We prove it by simple induction on the positive integers. Assume $0 \leq a \leq b$. The predicate $P(n)$ is $0 \leq a^n \leq b^n$. The base case is $n = 1$. By the assumption, $0 \leq a^1 \leq b^1$, so $P(1)$ is true. The induction hypothesis (IH) is that $P(k)$ is true for some $k \geq 1$. Thus we have

$$0 \leq a^k \leq b^k.$$  

Because $0 \leq a$, we may apply scaling invariance (Axiom 4.2.5) to these inequalities to get

$$0 \cdot a \leq a^k \cdot a \leq b^k \cdot a.$$  

Because $0 \leq b^k$, we may apply scaling invariance to $a \leq b$ to get

$$a \cdot b^k \leq b \cdot b^k.$$  

Because $a^{k+1} = a \cdot a^k$ and $b^{k+1} = b \cdot b^k$, we may apply transitivity (Axiom 4.2.3) to get

$$0 \leq a^{k+1} \leq b^{k+1},$$  

showing that $P(k+1)$ is true, which concludes the induction.

(d) If $a$ and $b$ are real numbers such that $0 \leq a \leq b$ then for all positive integers $m$ and $n$, if $m \leq n$ then $a^m \leq b^n$.

Solution. This is false. A counterexample is provided by $a = 1/4$, $b = 1/2$, $m = 1$ and $n = 3$. We have $a \leq b$ and $m \leq n$, but $a^m = 1/4$ while $b^n = 1/8$, so $a^m \leq b^n$ is false.

(e) If $a$ is a real number such that $-1 \leq a$ then for all positive integers $n$

$$1 + na \leq (1 + a)^n.$$  

Solution. This is true. We prove it by simple induction on the positive integers. Assume $a$ is a real number such that $-1 \leq a$. Then, adding 1 to both sides (translation invariance), we have $0 \leq (1 + a)$. The predicate $P(n)$ is $(1 + na) \leq (1 + a)^n$. The base case is $n = 1$, where $(1 + na) = (1 + a)$ and $(1 + a)^1 = (1 + a)$, so $P(1)$ is true. The induction hypothesis (IH) is that for some positive integer $k \geq 1$, $P(k)$ is true, that is,

$$1 + ka \leq (1 + a)^k.$$  

Because $0 \leq (1 + a)$, we may apply scaling invariance to get

$$(1 + a)(1 + ka) \leq (1 + a)(1 + a)^k = (1 + a)^{k+1}.$$  

3
Multiplying out the left hand side of this:

\[ 1 + (k + 1)a + ka^2 \leq (1 + a)^{k+1}. \]

However, \(0 \leq a^2\) and \(1 \leq k\), so \(0 \leq ka^2\), and by translation invariance we get

\[ 1 + (k + 1)a \leq 1 + (k + 1)a + ka^2, \]

so by transitivity of \(\leq\) we have

\[ 1 + (k + 1)a \leq 1 + (1 + a)^{k+1}, \]

that is, \(P(k + 1)\) is true, which concludes the induction.

4. (15 points) Prove from the definitions in Section 7.1 of the course text that if functions \(f\) and \(g\) are defined for every natural number \(n\) by \(f(n) = 3n^2\) and \(g(n) = n^3\) then \(f(n)\) is in \(O(g(n))\) but \(g(n)\) is not in \(O(f(n))\).

**Solution.** Let \(c = 3\) and \(N = 1\). Then for every \(n \geq 1\), by scaling invariance, \(n^2 \geq n\) and \(n^3 \geq n^2\) and \(3n^3 \geq 3n^2\). Because for every natural number \(n\), \(|n| = n\), we have for all \(n \geq N\)

\[ |f(n)| = 3n^2 \leq 3n^3 = 3|g(n)|, \]

we conclude that \(f(n)\) is in \(O(g(n))\).

To see that \(g(n)\) is not in \(O(f(n))\), let \(c\) and \(N\) be arbitrary real numbers such that \(c > 0\). Let \(n\) be any natural number greater than the maximum of \(3c\) and \(N\). Then, because \(n > 3c > 0\), by scaling invariance we have

\[ n^3 > 3cn^2. \]

Because \(|n| = n\) and \(c > 0\), we have

\[ |g(n)| > c|f(n)|. \]

This shows that there do not exist any constants \(c > 0\) and \(N\) such that for all natural numbers \(n \geq N\)

\[ |g(n)| \leq c|f(n)|, \]

that is, \(g(n)\) is not in \(O(f(n))\).

5. (15 points) Prove from the definitions in Section 7.1 of the course text that for all functions \(f\) and \(g\) with domain \(\mathbb{N}\) and co-domain \(\mathbb{Z}\), \(f(n)\) is in \(O(g(n))\) if and only if \(g(n)\) is in \(\Omega(f(n))\).

**Solution.** Assume that \(f(n)\) is in \(O(g(n))\). By the definition, this means that there exist constants \(c > 0\) and \(N\) such that for all natural numbers \(n \geq N\),

\[ |f(n)| \leq c|g(n)|. \]

Because \(c > 0\), \(1/c > 0\), and we may multiply both sides of this inequality by \(1/c\) to get

\[ (1/c)|f(n)| \leq |g(n)|. \]
Thus, there exists a constant $1/c > 0$ and a constant $N$ such that for all $n \geq N$,

$$(1/c)|f(n)| \leq |g(n)|,$$

which is the definition of $g(n)$ is in $\Omega(f(n))$.

Conversely, assume that $g(n)$ is in $\Omega(f(n))$. Then by definition, there exist constants $c > 0$ and $N$ such that for all $n \geq N$,

$$c|f(n)| \leq |g(n)|.$$ 

Because $c > 0$, $1/c > 0$, so we may multiply both sides of this inequality by $1/c$, which gives

$$|f(n)| \leq (1/c)|g(n)|.$$ 

Thus, there exist constants, $1/c > 0$ and $N$ such that for all $n \geq N$,

$$|f(n)| \leq (1/c)|g(n)|,$$

which is the definition of $f(n)$ is in $O(g(n))$.

6. (20 points) We define the function

$$q(n) = \sum_{i=1}^{n} i \cdot 5^i,$$

for all positive integers $n$. Prove that $q(n)$ is in $\Theta(n \cdot 5^n)$.

Solution. Note that in the sum, because $i \leq n$, each term $i \cdot 5^i$ is less than or equal to $n \cdot 5^i$. Thus

$$q(n) \leq \sum_{i=1}^{n} n \cdot 5^i,$$

and by linearity of summation, we have

$$q(n) \leq n \sum_{i=1}^{n} 5^i.$$

Using one of the three “standard sums” in 6.4.1, we have

$$\sum_{i=0}^{n} 5^i = (5^{n+1} - 1)/4.$$ 

Noting that $5^0 = 1$, we have

$$\sum_{i=1}^{n} 5^i \leq (5/4)5^n.$$ 

Thus for all $n \geq 1$,

$$q(n) \leq (5/4)(n \cdot 5^n),$$

which satisfies the definition of $q(n)$ is in $O(n \cdot 5^n)$ with $c = 5/4$ and $N = 1$.

Note that all of the terms of the summation are positive, so for all $n \geq 1$,

$$q(n) \geq n \cdot 5^n,$$

because the right hand side is the last term in the summation. This satisfies the definition of $q(n)$ is in $\Omega(n \cdot 5^n)$ with $c = 1$ and $N = 1$.

Because $q(n)$ is in $O(n \cdot 5^n)$ and also in $\Omega(n \cdot 5^n)$, we have that it is in $\Theta(n \cdot 5^n)$. 

5