SUBMISSION INSTRUCTIONS. Please turn in your homework in THREE parts, each part stapled together, with your name, the homework number and the problem numbers on the front of each part. The three parts will be placed in three separate labeled boxes:

• Problems 1 and 2
• Problems 3 and 4
• Problems 5 and 6

Please state (with problem 1) the names of any persons and references to any resources (including online) you consulted with in connection with this assignment, or state that none were consulted. (There will be 2 points off for failure to do so.)

For this assignment, while you may consult outside resources, any facts that you use in your proofs must be cited from the course text, Notes on Discrete Math, rather than outside resources.

1. (15 points) For each of the following binary relations \( R \) on the set \( A \), determine whether it has each of the following properties, and briefly justify your answer.

Relations:

(a) \( A \) is the positive integers and \( R \) is the set of pairs \((m, n)\) such that every prime that divides \( m \) also divides \( n \) and every prime that divides \( n \) also divides \( m \).

Solution. For every positive integer \( n \), we define \( P(n) \) to be the set of prime numbers that divide \( n \). Then \((m, n) \in R\) if and only if \( P(m) = P(n) \).

i. \( R \) is reflexive because for every positive integer \( n \), \( P(n) = P(n) \), so \((n, n) \in R\).

ii. \( R \) is symmetric because if \((m, n) \in R\) then \( P(m) = P(n) \) so \((n, m) \in R\).

iii. \( R \) is not antisymmetric because \( P(2) = \{2\} \) and \( P(4) = \{2\} \), so \((2, 4) \in R\) and \((4, 2) \in R\), but \( 2 \neq 4 \).

iv. \( R \) is transitive because if \((\ell, m) \in R\) and \((m, n) \in R\), then \( P(\ell) = P(m) \) and \( P(m) = P(n) \), so \( P(\ell) = P(n) \) and \((\ell, n) \in R\).

v. Because \( R \) is reflexive, symmetric and transitive, it is an equivalence relation.

(That \( R \) is an equivalence relation could also be proved by using condition (3) in Theorem 9.4.1 characterizing equivalence relations, applying it to the function \( P \) defined above.)

(b) \( A \) is all subsets of the nonnegative integers, i.e., \( A = \mathcal{P}(\mathbb{N}) \), and \( R \) is the set of pairs \((S, T)\) such that there exists a surjection \( f : S \to T \).

Solution.

i. \( R \) is reflexive. If \( S \) is any subset of \( \mathbb{N} \), the identity map \( f : S \to S \) defined by \( f(n) = n \) for all \( n \in S \) is a bijection (and therefore a surjection), so \((S, S) \in R\).

ii. \( R \) is not symmetric. To see this, let \( S = \{1, 2\} \) and \( T = \{1\} \). Then \((S, T) \in R\) (consider the surjection that maps both elements of \( S \) to 1) but \((T, S) \notin R\) (because the range of any function with domain \( T \) has exactly one element.)
2. (15 points) For each of the following sets of properties, determine whether there exists a
binary relation as a directed graph (NOT a Hasse diagram), and explain why the stated properties
are true of the relation. If not, prove that no such relation exists.

(a) \( R \) is reflexive, antisymmetric, and not transitive.

Solution. \( R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,c)\} \) is reflexive, antisymmetric and not
transitive. It is reflexive because \((x,x) \in R \) for all \( x \in \{a,b,c,d\} \). It is antisymmetric
because there do not exist \( x, y \in A \) such that \( x \neq y, (x,y) \in R \) and \( (y,x) \in R \). It is not
transitive because \( (a,b) \in R \) and \((b,c) \in R \) but \( (a,c) \notin R \).

(b) \( R \) is an antisymmetric equivalence relation.

Solution. \( R = \{(a,a), (b,b), (c,c), (d,d)\} \), the identity relation, is the only antisymmetric
equivalence relation on the set \( A = \{a,b,c,d\} \). It is reflexive because \((x,x) \in R \) for all
3. (20 points) Let \( R \) be a binary relation on a set \( A \). Recall that the composition of \( R \) with itself, denoted \( R \circ R \), is the binary relation on \( A \) consisting of the set of all pairs \((x, z)\) such that for some \( y \in A \), \((x, y) \in R \) and \((y, z) \in R \). Prove the following statements:

(a) If \( R \) is reflexive and transitive, then \( R = R \circ R \).

Solution. Assume \( R \) is reflexive and transitive. We prove \( R = R \circ R \) by proving \( R \subseteq R \circ R \) and \( R \circ R \subseteq R \). (1) Assume \((x, y) \in R \). Then because \( R \) is reflexive, \((y, y) \in R \). Using the definition of \( R \circ R \) on the pairs \((x, y) \in R \) and \((y, y) \in R \), we have \((x, y) \in R \circ R \). This shows \( R \subseteq R \circ R \). (2) Assume \((x, y) \in R \circ R \). Then by the definition of \( R \circ R \), there exists \( w \in A \) such that \((x, w) \in R \) and \((w, y) \in R \). Because \( R \) is transitive, this implies that \((x, y) \in R \). This shows \( R \circ R \subseteq R \).
(b) If \( R \) is antisymmetric and transitive, then \( R \circ R \) is also antisymmetric and transitive.

Solution. Assume \( R \) is antisymmetric and transitive.

To see that \( R \circ R \) is antisymmetric, assume that \((x,y)\) and \((y,x)\) are both in \( R \circ R \). By the definition of \( R \circ R \), this implies that there exists an element \( u \in A \) such that \((x,u) \in R \) and \((u,y) \in R \). Because \( R \) is transitive, this implies that \((x,y) \in R \). Similarly, there exists an element \( v \in A \) such that \((y,v) \in R \) and \((v,x) \in R \), and by the transitivity of \( R \), this implies \((y,x) \in R \). Since \((x,y)\) and \((y,x)\) are both in \( R \) and \( R \) is antisymmetric, this implies that \( x = y \), showing that \( R \circ R \) is antisymmetric.

To see that \( R \circ R \) is transitive, suppose \((x,y)\) and \((y,z)\) are in \( R \circ R \). By the definition of \( R \circ R \), this implies that there exist elements \( u, v \in A \) such that \((x,u) \in R \) and \((u,y) \in R \), and also \((y,v) \in R \) and \((v,z) \in R \). Because \( R \) is transitive, this implies that \((x,y) \in R \) and \((y,z) \in R \). Using the definition of \( R \circ R \), this implies that \((x,z) \in R \circ R \), which shows that \( R \circ R \) is transitive.

4. (15 points) Let \( A \) be the set of all nonempty finite strings of lower-case letters of the English alphabet. Thus, \( A \) contains strings \( app, apple, application \) and \( zzyyv \), among infinitely many others. Define the relation \( R \) on \( A \) to be the set of all pairs of strings \((s,t)\) such that either \( s = t \) or, if they are different, then \( s \) would precede \( t \) in a dictionary, with the usual alphabetic ordering of the letters. Thus, \((app,apple) \in R \), and \((apple,application) \in R \), but \((apple,apparition) \notin R \).

(a) Briefly describe an algorithm that takes as input two strings \( s, t \in A \), and returns 1 if \((s,t) \in R \) or 0 if \((s,t) \notin R \).

Solution. Let \( s \) and \( t \) be the input strings, of lengths \( \ell(s) \) and \( \ell(t) \), respectively. We index the letters of \( s \) from 1 to \( \ell(s) \) and the letters of \( t \) from 1 to \( \ell(t) \). The algorithm \( A \) initializes \( i = 1 \) and while \( i \) is less than or equal to both \( \ell(s) \) and \( \ell(t) \), it compares the letters \( s[i] \) and \( t[i] \). If \( s[i] \) precedes \( t[i] \) in the alphabet, then it halts and outputs 1. If \( t[i] \) precedes \( s[i] \) in the alphabet, then it halts and outputs 0. Otherwise, \( s[i] = t[i] \) and it increases \( i \) by 1 and repeats the while loop.

If the while loop terminates, then all the letters compared have been equal. If \( \ell(s) \leq \ell(t) \), then either \( s = t \) or \( s \) is a proper prefix of \( t \), so \( A \) halts and outputs 1. Otherwise, \( t \) is a proper prefix of \( s \), so \( A \) halts and outputs 0.

(b) Prove that \( R \) is a total order, that is, \( R \) is reflexive, antisymmetric and transitive, and any pair of strings \( s, t \in A \) are comparable, that is, \((s,t) \in R \) or \((t,s) \in R \).

We use the algorithm \( A \) described in part (a) to define the total ordering \( R \).

To see that \( R \) is reflexive, note that if \( s = t \) then the comparisons of all the letters will be equal and the lengths will be equal, so the algorithm will return 1, that is, \((s,s) \in R \).

To see that \( R \) is antisymmetric, assume that \( s \neq t \). If neither \( s \) nor \( t \) is a proper prefix of the other, then there exists some least index \( i \) such that \( s[i] \neq t[i] \). If \( s[i] \) precedes \( t[i] \) then \( A \) outputs 1 on the pair \((s,t)\) and 0 on the pair \((t,s)\). If \( t[i] \) precedes \( s[i] \) then \( A \) outputs 0 on the pair \((s,t)\) and 1 on the pair \((t,s)\). Otherwise, either \( s \) is a proper prefix of \( t \) or \( t \) is a proper prefix of \( s \). In the first case, \( A \) outputs 1 on the pair \((s,t)\) and 0 on the pair \((t,s)\). In the second case, \( A \) outputs 0 on the pair \((s,t)\) and 1 on the pair \((t,s)\). Thus, if \( s \neq t \), we cannot have both \((s,t) \in R \) and \((t,s) \in R \), so \( R \) is antisymmetric.

To see that \( R \) is transitive, assume \((s,t) \in R \) and \((t,u) \in R \). If \( s = t \) or \( t = u \) then \((s,u) \in R \), so assume \( s \neq t \) and \( t \neq u \). We consider four cases. (1) If \( s \) is a proper prefix
of $t$ and $t$ is a proper prefix of $u$, then $s$ is a proper prefix of $u$, so $A$ outputs 1 on $(s, u)$ and $(s, u) \in R$. (2) If $s$ is a proper prefix of $t$ and $t$ is not a proper prefix of $u$, then there exists a least index $i$ such that $t[i]$ precedes $u[i]$. If $i$ is less than or equal to the length of $s$, then on input $(s, u)$, $A$ will compare the letters through $i − 1$ and find them equal, and then find $s[i]$ (which equals $t[i]$) precedes $u[i]$ and output 1, so $(s, u) \in R$. If $i$ is greater than the length of $s$, then $s$ is a proper prefix of $u$, and $A$ outputs 1 on input $(s, u)$. In either case, $(s, u) \in R$. (3) If $s$ is not a proper prefix of $t$ and $t$ is a proper prefix of $u$, then there is a least index $i$ such that $s[i]$ precedes $t[i]$. Since $t$ is a prefix of $u$, $t[i] = u[i]$, so $A$ will compare letters of $s$ and $u$ up through $i − 1$ and find them equal, and then find that $s[i]$ precedes $t[i]$, and output 1 on input $(s, u)$, so $(s, u) \in R$. (4) If $s$ is not a proper prefix of $t$ and $t$ is not a proper prefix of $u$, then let $i$ be the least index such that $s[i]$ precedes $t[i]$ and let $j$ be the least index such that $t[j]$ precedes $u[j]$. If $i < j$, then $s$ and $u$ agree up to $i$, and $s[i]$ precedes $t[i] = u[i]$, so $A$ outputs 1 on input $(s, u)$. If $i = j$, then $s$ and $u$ agree up to $i$, and $s[i]$ precedes $t[i]$, which precedes $u[i]$, so $A$ outputs 1 on input $(s, u)$. If $i > j$, then $s$ and $u$ agree up to $j$, and $s[j] = t[j]$ and $t[j]$ precedes $s[j]$, so in this case also, $A$ outputs 1 on input $(s, u)$. Thus in all cases, $A$ outputs 1 on input $(s, u)$ so $(s, u) \in R$ and $R$ is transitive.

To see that $R$ is a total order, note that if $s \neq t$ then $A$ must output 1 on input $(s, t)$ or on input $(t, s)$, so $(s, t) \in R$ or $(t, s) \in R$, that is, $s$ and $t$ are comparable.

c) Find an infinite subset $S$ of $A$ that contains no minimal element (with respect to the total order $R$).

Solution. Consider the set of strings $S = \{yz, yyz, yyyy, yyyyz, \ldots \}$ consisting of a positive integer number of $y$’s followed by one $z$. Let $y^n z$ denote the string containing $n$ $y$’s followed by a $z$. To see that $S$ has no minimal element, consider any element $y^n z$ of $S$. Then $y^{n+1} z$ is also an element of $S$, but is smaller in the ordering $R$ (that is, $(xy^{n+1}, xy^n) \in R$) because when we compare $s = y^{n+1} z$ with $t = y^n z$, they are equal up to index $n + 1$ and $s[n + 1] = y$ and $t[n + 1] = z$. Since $y$ precedes $z$, $A$ outputs 1 on $(s, t)$, that is, $(s, t) \in R$. Thus, for every element of $S$ there is a smaller element of $S$, so $S$ contains no minimal element (with respect to $R$).
For reflexivity, since for any \( f \in A \), \( f(n) = f(n) \) for all \( n \geq 0 \), we have \( f \equiv^* f \).
For symmetry, if \( f \equiv^* g \) then there exists \( N \) such that for all \( n \geq N \), \( f(n) = g(n) \). Because = is symmetric, this means that for all \( n \geq N \), \( g(n) = f(n) \), so \( g \equiv^* f \). For transitivity, if \( f \equiv^* g \) and \( g \equiv^* h \) then there exist \( N_1 \) and \( N_2 \) such that for all \( n \geq N_1 \), \( f(n) = g(n) \) and for all \( n \geq N_2 \), \( g(n) = h(n) \). Taking \( N_3 = \max\{N_1, N_2\} \), for all \( n \geq N_3 \), \( f(n) = g(n) = h(n) \), so \( f \equiv^* h \).

(b) (5 points) For all \( f, g \in A \), we define \( f \leq^* g \) if and only if there exists a natural number \( N \) such that for all \( n \geq N \), \( f(n) \leq g(n) \). Prove that \( \leq^* \) is reflexive and transitive, but not antisymmetric.

**Solution.** For any \( f \in A \), \( f(n) \leq f(n) \) for all \( n \geq 0 \), so \( f \leq^* f \), and \( \leq^* \) is reflexive. Assume \( f \leq^* g \) and \( g \leq^* h \). Then there exist \( N_1 \) and \( N_2 \) such that for all \( n \geq N_1 \), \( f(n) \leq g(n) \), and for all \( n \geq N_2 \), \( g(n) \leq h(n) \). If we choose \( N_3 = \max\{N_1, N_2\} \), then for all \( n \geq N_3 \), \( f(n) \leq g(n) \leq h(n) \), so the transitivity of \( \leq \) implies that \( f(n) \leq h(n) \) for all \( n \geq N_3 \), and \( f \leq^* h \).

However, \( \leq^* \) is not antisymmetric. Define \( f(n) = 0 \) for all \( n \in \mathbb{N} \). Define \( g(n) = 0 \) for all positive integers \( n \) and \( g(0) = 17 \). Then, for all \( n \geq 0 \) we have \( f(n) \leq g(n) \), and \( f \neq g \) because \( f(0) = 0 \) and \( g(0) = 17 \).

(c) (10 points) Assuming the results in parts (a) and (b), we define for any \( f \in A \) the equivalence class of \( f \) as

\[
[f] = \{ g \in A \mid g \equiv^* f \}.
\]

Let \( A/\equiv^* \) denote the set of all equivalence classes of elements \( f \in A \). Then we define a binary relation on \( A/\equiv^* \) by

\[
R = \{ ([f], [g]) \mid f \leq^* g \}.
\]

Prove that \( R \) is well-defined and a partial order on \( A/\equiv^* \).

**Solution.** \( R \) is well-defined if and only if whether \( ([f], [g]) \) is in \( R \) does not depend on which representatives of \([f]\) and \([g]\) we choose to compare. That is, we must show that if \( f' \equiv^* f \) and \( g' \equiv g \) then \( f' \leq^* g' \) if and only if \( f \leq^* g \). Assume \( f' \equiv^* f \) and \( g' \equiv^* g \). Then there exist \( N_1 \) and \( N_2 \) such that for all \( n \geq N_1 \), \( f'(n) = f(n) \) and for all \( n \geq N_2 \), \( g'(n) = g(n) \). If \( f \leq^* g \) then there exists \( N_3 \) such that for all \( n \geq N_3 \), \( f(n) \leq g(n) \). Choose \( N_4 = \max\{N_1, N_2, N_3\} \). Then for all \( n \geq N_4 \),

\[
f'(n) = f(n) \leq g(n) = g'(n),
\]

which implies \( f' \leq^* g' \). The converse is completely symmetrical.

To see that \( R \) is a partial order on \( A/\equiv^* \), we must show it is reflexive, transitive, and antisymmetric. Let \( f \) be any element of \( A \). Because \( \leq^* \) is reflexive (part (b)), \( f \leq^* f \), so \( ([f], [f]) \in R \) and \( R \) is reflexive. If \( ([f], [g]) \in R \) and \( ([g], [h]) \in R \), then \( f \leq^* g \) and \( g \leq^* h \), and because \( \leq^* \) is transitive (by part (b)), this implies that \( f \leq^* h \), so \( ([f], [h]) \in R \), thus \( R \) is transitive.

To see that \( R \) is antisymmetric, we need to show that if \( ([f], [g]) \in R \) and \( ([g], [f]) \in R \) then \( [f] = [g] \). That is, we must show that if \( f \leq^* g \) and \( g \leq^* f \) then \( f \equiv^* g \). If \( f \leq^* g \) and \( g \leq^* f \) then there exist \( N_1 \) and \( N_2 \) such that for all \( n \geq N_1 \), \( f(n) \leq g(n) \) and for all \( n \geq N_2 \), \( g(n) \leq f(n) \). Taking \( N_3 = \max\{N_1, N_2\} \), for all \( n \geq N_3 \), \( f(n) \leq g(n) \leq f(n) \), so \( f \equiv^* g \). This implies that \( [f] = [g] \).
and for all $n \geq N_2$, $g(n) \leq f(n)$. Choosing $N_3 = \max\{N_1, N_2\}$, we have for all $n \geq N_3$, $f(n) \leq g(n)$ and $g(n) \leq f(n)$. By the antisymmetry of $\leq$, this implies that for all $n \geq N_3$, $f(n) = g(n)$, and therefore $f \equiv^* g$. 
