Economics and Computation

ECON 425/563 and CPSC 455/555

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Lecture X

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1 Combinatorial Auctions

1.1 Overview

Our treatment of combinatorial auctions will be organized as follows:

1. Ascending Auctions and the Vickrey-Clarke-Groves (VCG) mechanism;
2. Linear programming techniques and the concept of Walrasian equilibrium;
3. Approximation and communication complexity.

1.2 Basic Elements and Notation

- The **set of objects/items** is given by $\mathcal{K} = \{1, ..., K\}$. The **set of possible bundles** that can be formed from K is denoted by the power-set[1] $2^{\mathcal{K}}$. An arbitrary bundle is denoted by $S \in 2^{\mathcal{K}}$. Observe that the cardinality of $2^{\mathcal{K}}$, i.e. the number of elements in the set $2^{\mathcal{K}}$, is given by 2^K.
- The **set of bidders/agents** is given by $\mathcal{N} = \{1, ..., N\}$.
- Agent $n \in \mathcal{N}$ has a **valuation function**

$$v_n : 2^{\mathcal{K}} \to \mathbb{R}_+.$$

  That is, each agent $n \in \mathcal{N}$ assigns a non-negative number to every possible subset of the set of objects $\mathcal{K}$.

Most importantly, the valuation-function v_n is agent n's private knowledge.[2] She can be asked to report it, but it is by no means clear that agent n will actually tell the truth. So, it is the task of the mechanism-designer to set up a mechanism such that it is in the agent's self-interest to report her valuation. This will be a crucial feature of mechanisms that we will be talking about below.

In order to reflect different preferences for bundles of items, certain restrictions can be imposed on the valuation-function v_n for an agent $n \in \mathcal{N}$:

- v_n can be assumed to be **additive**, i.e.

$$v_n(S) = \sum_{k \in S} v_n(k).$$

[1]The power-set of $\mathcal{K}$ refers to the set of all subsets of $\mathcal{K}$. For example for the set $\{1, 2, 3\}$ the power-set is given by $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

[2]So, the agent's valuation function will be regarded as his type. This has previously been denoted by t_n.

Under the linearity-assumption, one obtains the valuation for a bundle by simply adding up the valuations for the objects in $\mathcal{K}$ that constitute the subset S. In consequence, one does not need to worry about specifying the valuation of agent n for any possible subset $S \subseteq K$, but all valuations can be reverse-engineered from the valuations of the objects $k \in \mathcal{K}$.
Another important aspect is related to the point of view of an auctioneer. She does not have to worry about offering bundles of the objects to be auctioned off, but it is sufficient for her to assign each object $k \in \mathcal{K}$ in a separate auction.

- Alternatively, the valuation function might be defined in a non-additive manner for two arbitrary subsets $S, T \in 2^{\mathcal{K}}$ satisfying $S \cap T = \emptyset$:
  - S and T are said to be **complements** iff
  $$v_n(S \cup T) \geqslant v_n(S) + v_n(T).$$
  That is, agent n values having both bundles more than getting either one of them.
  - S and T are said to be **substitutes** iff
  $$v_n(S \cup T) \leqslant v_n(S) + v_n(T).$$
  That is, agent n values having both bundles less than getting either one of them.

Making use of the valuation-function[3] v_n of any agent $n \in \mathcal{N}$, it will be assumed that **agent** n**'s utility function** has the following functional form:

$$\begin{aligned} u_n : 2^{\mathcal{K}} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (S, t) &\mapsto v_n(S) - t \end{aligned}$$

So, agent n's utility has an arbitrary subset $S \subseteq \mathcal{K}$ and a **monetary transfer** t as an input. Then, her utility is given by her valuation of the bundle S minus the transfer that she has to make. This particular specification of the utility-function is called **quasi-linear utility**.

In order to complete the model, one also needs to think about the assignment of a utility-function to the auctioneer, which we will also refer to as the government or the residual

[3] A priori, none of the properties for v_n discussed above will be assumed to hold. The following statements are valid for any functional assumption of $v_n : 2^{\mathcal{K}} \rightarrow \mathbb{R}_+$.

recipient.[4] This utility function, denoted by u_0, is given by the following mapping:

$$\begin{aligned} u_0 : 2^{\mathcal{M}} \times \mathbb{R}^N &\rightarrow \mathbb{R} \\ (S; t_1, ..., t_N) &\mapsto \sum_{n=1}^{N} t_n \end{aligned}$$

The government's utility function takes a subset from $\mathcal{M}$ and the transfers from all agents as an input. Its utility value is simply the sum of the transfers. The fact that a subset of $\mathcal{M}$ appears as an input of u_0, but does not affect the government's utility at all is purely due to notational convenience. It is by no means clear that the auctioneer will necessarily distribute all objects to the bidders. Hence, the government will be considered as an additional entity that will receive all unassigned objects. The subset of the set $\mathcal{K}$ that the government receives will be denoted by S_0. Furthermore, we will use the notation

$$\mathcal{N}_0 \triangleq \{0\} \cup \mathcal{N}$$

to describe the set of agents plus the government.

1.3 Efficiency

In this subsection, we take the point of view of the auctioneer and try to determine the "best" or efficient way (which will be made precise below) to distribute the M objects among the N bidders and the government, i.e. over the set $\mathcal{N}_0$.

In order to come to the concept of efficiency, we need to formally define the concept of a **(feasible) allocation**:[5]

Definition 1 *A **feasible allocation** is a partition of the set $\mathcal{K}$ over the set $\mathcal{N}_0$, i.e. a collection of subsets*

$$S = (S_0, S_1..., S_N)$$

such that:

1. *For all $n, n' \in \mathcal{N}_0$*
$$S_n \cap S_{n'} = \emptyset;$$
2. $$\bigcup_{n=0}^{N} S_n = \mathcal{K}.^6$$

[4]In the following analysis, we will always assume that the auctioneer is benevolent, i.e. he does not have a self-interest, but acts in the interest of the entity of the participants in the auction.

[5]We will not consider any unfeasible allocations. Therefore, the following definition of "allocation" already incorporates feasibility.

[6]It is exactly the inclusion of the government that yields equality in this condition.

The ***set of all such partitions*** *(feasible allocations) will be denoted by* $\mathcal{S}$.

So, an allocation assigns a subset of objects from $\mathcal{K}$ to each player $n \in \mathcal{N}$. We require the intuitive to be satisfied that no object is assigned to more than one player (property 1.) and that the union of all assignments stays within the bounds of the set of available objects (property 2.)

In order to obtain efficiency, it is necessary to aggregate all market-participants' utilities into a social (economy-wide) utility. An example of this aggregation is to take sum all utilities, to which we will restrict attention in the following:[7]

Taking an arbitrary feasible allocation $(S_0, S_1, ..., S_N)$ and a tuple of transfer $t_1, ..., t_N$, the social utility is given by

$$\begin{aligned} & \sum_{n=1}^{N} u_n(S_n, t_n) + u_0(S_0; t_1, ..., t_N) \\ = & \sum_{n=1}^{N} [v_n(S_n) - t_n] + \sum_{n=1}^{N} t_n \\ = & \sum_{n=1}^{N} v_n(S_n) \end{aligned}$$

The last expression (the sum of all agent's valuations) is also referred to **gross utility**.

So, an **efficient allocation** is a feasible allocation that maximizes gross or social utility, i.e. $(S_1^*, ..., S_N^*)$ is called an efficient allocation iff

$$(S_1^*, ..., S_N^*) \in \operatorname{argmax} \left\{ \sum_{n=1}^{N} v_n(S_n) \right\} \quad \text{s.t. feasibility-conditions 1. and 2.}$$

Previously, we have already discussed the Second-Price auction as an example of an efficient allocation.[8] In the following, we aim to generalize the concept of the Second-Price auction by introducing the Vickrey-Clarke-Groves[9] (VCG) mechanism. But, before that, we have to clarify the notions of **direct mechanism** and **truthful revelation in dominant strategies**.

[7]The sum of all participants' utilities is by far the most widespread criterion for efficiency.

[8]Our argument for efficiency rested on the fact that the agent/bidder with the highest valuation received the object

[9]The corresponding articles are Vickrey (1962), Clarke (1972) and Groves (1974).

1.4 Valuation-Functions and Construction of a Direct Mechanism

A **direct mechanism** is given by the following pair of mappings:

- An allocation:

$$a : \mathbb{R}_+^{2^K \cdot N} \to \mathcal{S}$$

  $a = (a_1, ..., a_N)$, where $a_n : \mathbb{R}^{2^K \cdot N} \to 2^{\mathcal{K}}$ denotes the allocation that is assigned to player $n \in \mathcal{N}$.

- A vector of transfers:

$$t : \mathbb{R}_+^{2^K \cdot N} \to \mathbb{R}^N$$

  .

Both mappings that constitute a direct mechanism have $\mathbb{R}_+^{2^K \cdot N}$ as their domain. That, is they take a report about all valuations (remember that an agent's valuations specifies a non-negative value for every subset, i.e. a valuation-vector for an agent has length $\mathbb{R}_+^{2^K}$) from all agents (N agents) as their input. Then, the allocation-mapping outputs a (feasible) allocation as described above and the transfer-mapping specifies a monetary amount that agent n, $n \in \mathcal{N}$, has to pay to the government (the benevolent planner).

1.5 Design of the Transfer

The transfer-vector $t = (t_1, ..., t_N) \in \mathbb{R}^N$ will be specified with a very particular goal in mind:

Truth-telling shall be a dominant strategy for the agent, i.e.

For all $n \in \mathcal{N}$ and all $v_n \in \mathbb{R}_+^{2^K}$ the following condition is satisfied

$$\begin{aligned} & v_n(a_n(v_1', ..., v_n, ..., v_N')) - t_n(v_1', ..., v_n, ..., v_N') \geqslant \\ & v_n(a_n(v_1', ..., v_n', ..., v_N')) - t_n(v_1', ..., v_n', ..., v_N') \qquad \forall \text{ tuples } (v_1', ..., v_n', ..., v_N') \end{aligned}$$

The following remarks on this condition can be made:

- Player n compares two different regimes. In the first regime, she truthfully reports the $\mathbb{R}_+^{2^K}$-vector to the mechanism-designer as v_n. In the second regime, she makes up a valuation-vector v_n' to report to the mechanism-designer.

- Agent n makes the comparison between the two scenarios for all possible reports that the other agents may submit, i.e. for $(v_1', ..., v_{n-1}', v_{n+1}', ..., v_N')$. This is the characteristic feature of the concept of "dominant strategies", because these strategies are optimal irrespective of the opponents' actions. This is in contrast to any notion of Nash-equilibrium that we have considered so far. Here, one presupposes a certain kind of action for the opponents, namely the equilibrium-actions.

- The vectors of valuations for all agents are plugged into the allocation-function and agent n's allocations as represented by the nth row a_n of the allocation-matrix.

- Finally, the allocation that agent n is assigned is evaluated according to agent n true valuation function v_n.

Abstractly, agent n's strategy, $n \in \mathcal{N}$, in this setting can be defined as a mapping r_n as follows:

$$m_n : \mathbb{R}_+^{2^K} \rightarrow \mathbb{R}_+^{2^K}.$$

So, agent n takes her valuation-vector v_n and transforms it into her report (which is often referred to as her message) $m_n(v_n)$. It will only be her report that she will announce toward the mechanism-designer. In contrast, v_n will remain her private knowledge.
The concept of truthful revelation corresponds to r_n being the identity-mapping.

1.5.1 VCG-mechanism

As a particular example of a transfer-scheme which induces truth-telling as a dominant strategy, the VCG-mechanism will be specified in the following. This mechanism is also referred to as **social externality pricing**. That is, any agent is supposed to make a payment according to the negative externality that she imposes on the remaining agents by her presence. In the following, we will, for notational convenience, adhere to the originally introduced notion of the valuation-function that has the power-set $2^{\mathcal{K}}$ as its domain.

Compare the following two social programs:[10]

- Social program including agent j:

$$S^* = (S_1^*, ..., S_{j-1}^*, S_j^*, S_{j+1}^*, ..., S_N^*) \in \text{argmax} \sum_{n=1}^{N} v_n(S_n).$$

- Social program excluding agent j:

$$S_{-j}^* = (S_1^*, ..., S_{j-1}^*, S_{j+1}^*, ..., S_N^*) \in \text{argmax} \sum_{n \neq j} v_n(S_n).$$

[10]Both programs are obviously computed under the feasibility-restriction.

In the social program including agent j, the objects from the set $\mathcal{K}$ are optimally and feasibly distributed taking all agents into account. In contrast, the second program simply excludes agent j from the determination of an optimal and feasible distribution of the elements in $\mathcal{K}$.

Denote

$$\begin{aligned} V^* &\triangleq \sum_{n=1}^{N} v_n(S_n^*), \\ V_{-j}^* &\triangleq \sum_{n \neq k} v_n(S_{-j,n}^*). \end{aligned}$$

So, V^* and V_{-j}^* describe the levels of social utility from the two social programs.

Observe the following general properties that arise from the comparison of the two programs:

1. Assume $S_j^* = \emptyset$. That is, in the program that includes agent j, this particular agent optimally receives no element from $\mathcal{M}$. In other words, agent j does not impose any externality on the other agents. In this case, the difference between V^* and V_{-j}^* is equal to zero.

2. In general, the following inequality holds:

$$V^* \geqslant V_{-j}^*.$$

   The optimization-problem for V_{-j}^* can be seen as a special case of the optimization-problem for V^* in which the additional restriction $S_j^* = \emptyset$ is imposed. So, the inequality above simply originates from the fact that the maximum for V^* is taken over a superset of the set over which V_{-j}^* is determined.

Now, we are in the position to ask ourselves what is the exact amount of the externality that agent j should be charged. It is flawed to simply take the difference between V^* and V_{-j}^*, because this ignores agent j's contribution to social welfare in the first program. Part of the payment of agent j would be his own valuation that he contributes to social welfare. This problem can be overcome by the following definition of the transfer in the VCG-mechanism:

$$t_j^{\text{VCG}} \triangleq \sum_{n \neq j} v_n(S_{-j,n}^*) - \sum_{n \neq j} v_n(S_n^*).$$

This exactly reflects the notion of social externality pricing. Agent j is charged the difference in the sum of the utilities of all other agents (social utility without agent j), when she is not considered in the allocation (first sum) and when she is (second sum).

Lemma 1

For all $j \in \mathcal{N}$, the inequality $t_j^{VCG} \geqslant 0$ holds.

Proof of Lemma 1

The first summand $\sum_{n \neq j} v_n(S^*_{-j,n})$ in the definition of t_j^{VCG} describes the optimal level of social utility that can be achieved of one takes agents $(1, ..., j-1, j+1, ..., N)$ into account. The second summand simply denotes another level of social utility in the situation that considers $(1, ..., j-1, j+1, ..., N)$. Therefore, the second summand is necessarily smaller than or equal to the first summand.

□

Making use of the definition of t_j^{VCG}, agent j's utility from the VCG-mechanism is given by

$$v_j(S^*_k) - t_j^{\text{VCG}}.$$

Lemma 2

The following two properties hold for any agent $k \in \mathbb{N}$:

1. *Agent j will participate in the mechanism.*[11] *Put differently, her utility is non-negative, i.e.*
$$v_j(S^*_j) - t_j^{VCG} \geqslant 0.$$
2. *Truth-telling is a dominant-strategy.*

Proof of Lemma 2

Applying the definition of t_j^{VCG}, one obtains

$$\begin{aligned} v_j(S^*_j) - t_j^{\text{VCG}} &= v_j(S^*_j) - \left[\sum_{n \neq j} v_n(S^*_{-j,n}) - \sum_{n \neq j} v_n(S^*_n) \right] \\ &= v_j(S^*_j) + \sum_{n \neq j} v_n(S^*_n) - \sum_{n \neq j} v_n(S^*_{-j,n}) \\ &= \sum_{n=1}^{N} v_n(S^*_n) - \sum_{n \neq j} v_n(S^*_{-j,n}) && (1) \\ &= V^* - V^*_{-j}. && (2) \end{aligned}$$

As it has been argued above in the general property 2., the difference $V^* - V^*_{-j}$ from (2) is non-negative, proving part 1. of the claim.

[11] Implicit in this statement is the assumption that the outside-option of any agent is zero, i.e. any agent who does not participate in the mechanism obtains zero utility.

In order to verify claim 2., observe that the expression $V^*_{-j} = \sum_{n \neq j} v_n(S^*_{-j,n})$ in (1) does not depend on the report of agent j, because he is simply not taken into account. So, agent j tries to choose a report in order to maximize the social utility $V^* = \sum_{n=1}^{N} v_n(S^*_n)$ in (1). This will guarantee herself maximum-possible utility. But fixing the other agents' reported valuations, it is exactly the true valuation for agent j which will yield the best possible level of social utility for agent j, so it is optimal for agent j to tell the truth. Because the other agents' reports have been assumed to be arbitrary, it follows that truth-telling is a dominant strategy for agent j.

□

1.6 Special Cases of the VCG-Mechanism

In this part we will look at the VCG-mechanism in the specific context of auctions with unit demand.[12] We will assume that all bidders are sorted by their valuation, i.e. we have

$$v_1 \geqslant v_2 \geqslant \ldots \geqslant v_N.$$

So, first consider a situation in which there is only one good to be auctioned off:

- Each agent/bidder simultaneously submits a bid for an object.
- The person with the highest bid wins the object.

So, what will be the transfers that the VCG-mechanism prescribes? Remember that VCG implies that truth-telling is a dominant strategy, so we do not have to care about strategies, but can simply restrict attention to valuations of the agents:

- For agent 1, i.e. $j = 1$:
  - According to the rules of the auction, she will be the bidder who receives the object.
  - Social utility is $V^* = v_1$ in the program that includes her, since she is the only one who receives the object.
  - If agent 1's valuation is subtracted from v^* in order to obtain the sum $\sum_{n \neq j} v_n(S^*_n)$, it follows that

$$\sum_{n \neq j} v_n(S^*_n) = 0.$$

[12]Unit demand means that each bidder/agent only wants to obtain one object.

  - Now, assume that agent 1 is excluded from the social program. Then, it will be agent 2 who receives the only object to be auctioned off. In consequence, social utility equals agent 2's valuation, i.e.

$$\sum_{n \neq j} v_n(S^*_{-j,n}) = v_2.$$

  - By the definition of t_1^{VCG}, it follows that

$$t_1^{\text{VCG}} = v_2.$$

- For any agent $j \in \{2, ..., N\}$:
  - According to the rules of the auction, she will not receive the object, so her valuation is zero.
  - Social utility is $v^* = v_1$ in the program that includes her.
  - Because agent j's valuation is zero, v^* remains unchanged if agent j's valuation is subtracted. Therefore, it follows that

$$\sum_{n \neq j} v_n(S^*_n) = v_1.$$

  - Now, assume that agent j is excluded from the social program. Then, it will still be agent 1 who receives the only object to be auctioned off. In consequence, social utility equals agent 1's valuation, i.e.

$$\sum_{n \neq j} v_n(S^*_{-j,n}) = v_1.$$

  - By the definition of t_j^{VCG}, it follows that

$$t_j^{\text{VCG}} = 0.$$

Therefore, it is only agent 1 who has to make a payment and this payment equals the second-highest bid/valuation. This is exactly the logic of the Second-Price auction.

Now, consider a situation in which there are k identical goods to be auctioned off:[13]

- Each agent/bidder simultaneously submits a bid for an object.

[13] For ease of exposition, we will assume that the number of bidders is strictly bigger than the number of objects to be auctioned off.

- The person with the m highest bids each win one object.

So, what will be the transfers that the VCG-mechanism prescribes? Again, remember that VCG implies that truth-telling is a dominant strategy, so we do not have to care strategies, but can simply restrict attention to valuations of the agents:

- For any agent $j \in \{1, ..., k\}$:
  - According to the rules of the auction, she will one of the bidders who receive one object.
  - Social utility is $V^* = \sum_{n=1}^{k} v_n$ in the program that includes her.
  - If agent j's valuation is subtracted from v^* in order to obtain the sum $\sum_{n \neq j} v_n(S_n^*)$, then it follows that

$$\sum_{n \neq j} v_n(S_n^*) = \sum_{n \in \{1,...,k\} \setminus \{j\}} v_n.$$

  - Now, assume that agent j is excluded from the social program. Then, it will be agents $1, ..., j-1, j+1, ..., k, k+1$ who receive one object to be auctioned off. In consequence, social utility equals

$$\sum_{n \neq j} v_n(S_{-j,n}^*) = \sum_{n \in \{1,...,k+1\} \setminus \{j\}} v_n.$$

  - By the definition of t_j^{VCG}, it follows that

$$t_j^{\mathrm{VCG}} = v_{k+1}.$$

- For any agent $j \in \{k+1, ..., N\}$:
  - According to the rules of the auction, she will not receive the object, so her valuation is zero.
  - Social utility is $v^* = \sum_{n=1}^{k} v_n$ in the program that includes her.
  - Because agent j's valuation is zero, v^* remains unchanged if agent j's valuation is subtracted. Therefore, it follows that

$$\sum_{n \neq j} v_n(S_n^*) = \sum_{n=1}^{k} v_n.$$

  - Now, assume that agent j is excluded from the social program. Then, it will still be agents $1, ..., k$ who receive one unit of the object to be auctioned off. In consequence, social utility equals the sum of agent 1's to k's valuation, i.e.

$$\sum_{n \neq j} v_n(S_{-j,n}^*) = \sum_{n=1}^{k} v_n.$$

– By the definition of t_j^{VCG}, it follows that

$$t_j^{\text{VCG}} = 0.$$

Therefore, it is agents $1, ..., k$ who have to make a payment and these payments equal the (k+1)-highest bid/valuation. This is a generalization of the notion of a Second-Price auction, which is called $(k+1)$**th Price auction**.