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Economics and Computation

Computer Science 455/555 and Economics 425/563
Solutions to Exam 1

1. Bayesian Game.

   (a) Give a general description of a Bayesian game (*i.e.* a game with incomplete information) and the notion of a pure strategy in a Bayesian game.
   We defined a Bayesian game as a tuple

$$\Gamma_B = \left\{\mathcal{I}, \{A_i\}_{i\in\mathcal{I}}, \{T_i\}_{i\in\mathcal{I}} \{u_i\}_{i\in\mathcal{I}}, p\right\}$$

   where $\mathcal{I} = \{1, ..., I\}$ is the set of players and A_i is the set of pure actions, T_i the set of types and $p(t)$ the common prior probability distributions over types. A pure strategy Bayesian Nash equilibrium is $s_i : T_i \to A_i$.

   (b) Define the notion of Bayes-Nash equilibrium. Define the notion of an ex post (Bayes-)Nash equilibrium and of a (Bayes-) Nash equilibrium in dominant strategies. (In all cases, it is sufficient to describe the notion of a pure strategy, and you do not have to describe the mixed strategy version).
   A Bayes-Nash equilibrium is $s^*(t) = (s_1^*(t_1), ..., s_I^*(t_I))$ such that

$$\sum_t p(t)\, u_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), (t_i, t_{-i})\right) \geq \sum_t p(t)\, u_i\left(\left(s_i'(t_i), s_{-i}^*(t_{-i})\right), (t_i, t_{-i})\right),$$

   for all i and all $s_i'(t_i)$.
   An ex post Bayes-Nash equilibrium is $s^*(t) = (s_1^*(t_1), ..., s_I^*(t_I))$ such that

$$u_i\left(\left(s_i^*(t_i), s_{-i}^*(t_{-i})\right), (t_i, t_{-i})\right) \geq u_i\left(\left(a_i, s_{-i}^*(t_{-i})\right), (t_i, t_{-i})\right),$$

   for all i and all a_i and all t.
   A Bayes-Nash equilibrium in dominant strategies is $s^*(t) = (s_1^*(t_1), ..., s_I^*(t_I))$ such that

$$u_i\left(\left(s_i^*(t_i), a_{-i}\right), (t_i, t_{-i})\right) \geq u_i\left(\left(a_i, a_{-i}\right), (t_i, t_{-i})\right),$$

   for all i and all a and all t.

   (c) Briefly discuss the main differences between these equilibrium notions and their relationship to each other.
   We have the following relationship

$$BNE_d \subseteq BNE_{ep} \subseteq BNE$$

   and in most games the inclusion is strict. The Bayes Nash equilibrium is requiring optimality in expectation, the ex post Bayes-Nash equilibrium requires optimality with respect to the equilibrium strategy and all possible type profile realization, and finally the Bayes Nash equilibrium in dominant strategy requires optimality irrespective of the type and the actions of all of the other players.

2. Consider the following single unit auction, often called an all pay auction. There are two bidders, $i \in \{1,2\}$ and the valuation v_i of each bidder is private information. The common prior distribution is identical and independently distributed by $v_i \sim \mathcal{U}[0,1]$. Each bidder is asked to submit a bid b_i (knowing his own valuation but not the valuation of his opponent). The rules of the auction are that the highest bidder receives the object, but that each bidder has to pay his own bid, irrespective of winning or loosing. (If by chance, the two bidders submit the same bid, then the assignment of the object is determined by chance with equal probability for each agent).

   (a) Carefully describe the ex post payoff function, *i.e.* the net utility of each bidder (as a function of his type, his bid and the bid of his opponent). Define a pure strategy for each player.

   The ex post utility is given by

$$u_i(v_i, b_i, b_j) = \begin{cases} v_i - b_i & \textbf{if } b_i > b_j, \\ \frac{1}{2}v_i - b_i & \textbf{if } b_i = b_j, \\ -b_i & \textbf{if } b_i < b_j, \end{cases}$$

   and a strategy is

$$b_i : [0,1] \to [0,1] \, .$$

   (b) Does there exist a dominant strategy for each agent in the all pay auction, argue carefully for or against.

   No. Suppose there exists one, then it could not be $b_i(v_i) = 0$ for all $v_i > 0$, because if the bid $b_j > 0$ of the opponent were lower than v_i, then v_i would like to make a bid a bit above b_j. But on the other hand, bidder i would only like to make a zero bid if the opponent bids above v_i, hence there does not exists a dominant strategy for bidder i.

   (c) Derive a Bayes Nash equilibrium with symmetric strategies of the form $b_i(v_i) = cv_i^2$; in particular determine the value of c.

   If agent i bids b_i, he get

$$v_i \Pr(b_i > b_j) - b_i$$

   and inserting we get

$$v_i \sqrt{\frac{b_i}{c}} - b_i$$

   and the first order condition is

$$\frac{1}{2} v_i \sqrt{\frac{1}{b_i c}} - 1 = 0$$

   or

$$\frac{1}{c}\left(\frac{1}{2}v_i\right)^2 = b_i$$

   and hence by symmetry

$$\frac{1}{c}\left(\frac{1}{2}\right)^2 = c$$

   or

$$c = \frac{1}{2}.$$

3. Consider the following version of a job scheduling problem as a combinatorial auction. There are three bidders, each bidder i needs to have a computing job completed at a central computing facility. Each single computing job takes 24 hours and thus at any given day at most one job can be started and completed. The computing facility does not allow parallel computing. The value of the job for bidder i is

$$v_i \cdot \delta^t$$

where $\delta \in (0,1)$, and $t = 1, 2$ is the number of days that bidder i has to wait to see his computing job scheduled. Thus if the job of bidder i is accepted today, $t = 0$, and if it is accepted tomorrow then $t = 1$, and so on... . The value of each computing job v_i is private information to bidder i.

(a) Suppose that the social planner would like to maximize the sum of the utilities, adjusted for the discount factor δ^t, *i.e.*

$$\delta^0 \cdot v_k + \delta^1 \cdot v_l + \delta^2 \cdot v_m$$

What would the efficient order of the jobs be so that the social welfare is maximized (if the social planner were to know the true valuations of the bidders).

Since $\delta \in (0,1)$, it will be efficient to rank the alternatives so that $v_1 > v_2 > v_3$ and the job with the higher valuation would be scheduled first.

(b) Suppose the bidders all arrive at $t = 0$ and are asked to report their valuations. Define the Vickrey-Clarke-Groves payments scheme in $t = 0$, which would induce the bidders to report their valuation truthfully in a dominant strategy. Explain in a few words the nature of the transfer price and how it relates to the social externality of each bidder.

The transfer payment would be

$$\begin{aligned} p_1 &= v_2 + \delta (v_3 - v_2) + \delta^2 (-v_3) \\ p_2 &= \delta v_3 + \delta^2 (-v_3) \\ p_3 &= 0 \end{aligned}$$

The payment reflects the opportunity cost, which is the direct opportunity cost less the improvement in future allocations which comes from the presence of a highly valuable job.

4. (a) In the AS graph in Figure 1, directed edges point from customers to providers, and undirected edges are peer edges. The destination node is d, and the source nodes are 1, 2, 3, 4, 5, and 6. All source nodes export all routes they know about to all of their neighbors.

Route-selection policies are such that ASes 1, 4, 5, and 6 value *shortest* routes to the destination more highly than all others, AS 2 values customer routes more highly than all others, and AS 3 values peer routes more highly than all others.

Identify a dispute wheel in this interdomain-routing instance. You need not specify the wheel formally in terms of the relations Θ_1 and Θ_2 in Chapter 14. Simply identify the nodes that are in dispute and explain why their route-selection policies lead to a dispute.

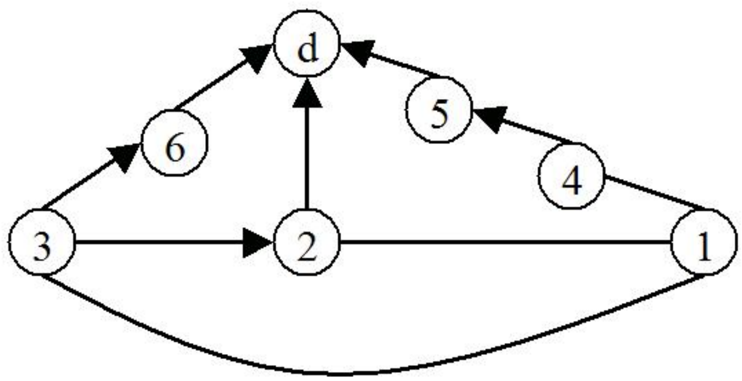
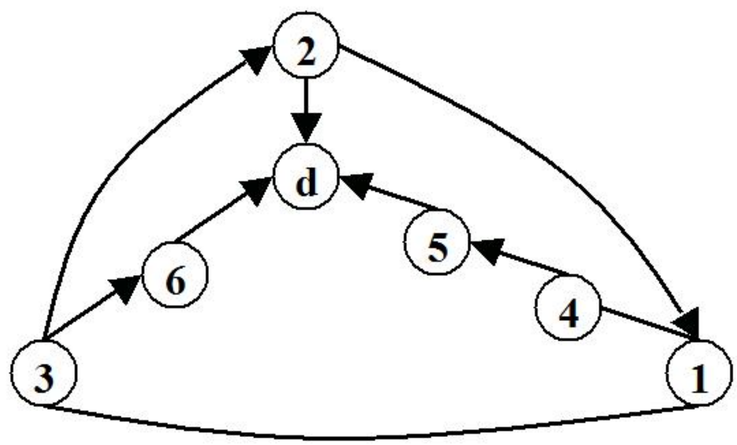


Figure 1

The dispute wheel has 1, 2, and 3 around the "rim" d in the middle, spoke "routes $1-4-5-d$, $2-d$, and $3-6-d$. The nodes involved in the dispute are 1, 2, and 3. 1 prefers to route through 2, because $1-2-d$ is shortest. 2 prefers to route through 3, because 3 is its only customer. 3 prefers to route through 1, because 1 is its only peer.



Solution Figure 1:
Dispute wheel

(b) In the AS graph in Figure 2, directed edges point from customers to providers, and undirected edges are peer edges. The destination is d.

Suppose that the route-export policies of source nodes 1, 2, and 3 obey the Gao-Rexford constraints. Suppose further that, under normal operating conditions, AS 2 uses the route $2-d$, and AS 3 uses $3-d$. Why would AS 2 be able to switch to $2-1-d$ if the link $2-d$ goes down but AS 3 not be able to switch to $3-1-d$ if the link $3-d$ goes down?

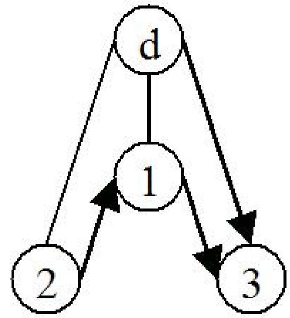


Figure 2

According to the Gao-Rexford "scoping" constraints, AS1 will export its peer route $1 - d$ to its customer AS 2 but not to its provider AS 3.

(a) Figure 3 shows a selfish-routing instance in which one unit of flow ($r = 1$) is to be routed from s to t. What is the price of anarchy in this instance?

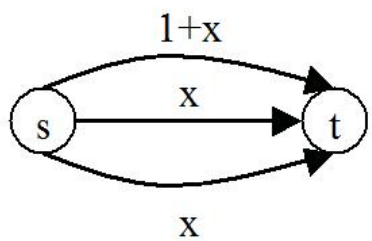


Figure 3

In the equilibrium flow f, a flow of 1/2 goes across each edge with cost x. The cost of these paths is then 1/2, which is less than the cost, 1, of the third path. This satisfies the equilibrium conditions. The cost of f is $C(f) = (1/2)(1/2) + (1/2)(1/2) = 1/2$.

The optimal flow f^* is identical to the equilibrium flow. The marginal cost of the paths of cost x is then $2(1/2) = 1$. The marginal cost of the path with cost function $1 + x$ is 1. This satisfies the conditions for the optimum. The cost of f^* is $C(f^*) = 1/2$.

Therefore the price of anarchy is 1.

(b) Recall the network-formation game of AGT, §19.2:

Players in the local connection game are identified with nodes in a graph G on which the network is to be built. A strategy for player u is a set of undirected edges that u will build, all of which have u as one endpoint. Given a strategy vector S, the set of edges in the union of all players' strategies form a network $G(S)$ on the player nodes. Let $dist_S(u, v)$ be the length of the shortest path (in terms of number of edges) between u and v in $G(S)$. The cost of building an edge is specified by a single parameter α. Each player seeks to make the distances to all other nodes small and to pay as little as possible. More precisely, player u's objective is to minimize the sum of costs and distances $\alpha n_u + \sum_v dist(u, v)$, where n_u is the number of edges bought by player u. We say that a network $G = (V, E)$ is *stable* for a value α if there is a stable strategy vector S that forms G.

Prove that the graph of Figure 4 is not stable, *i.e.*, is not a Nash Equilibrium of this game, regardless of the edge cost α.

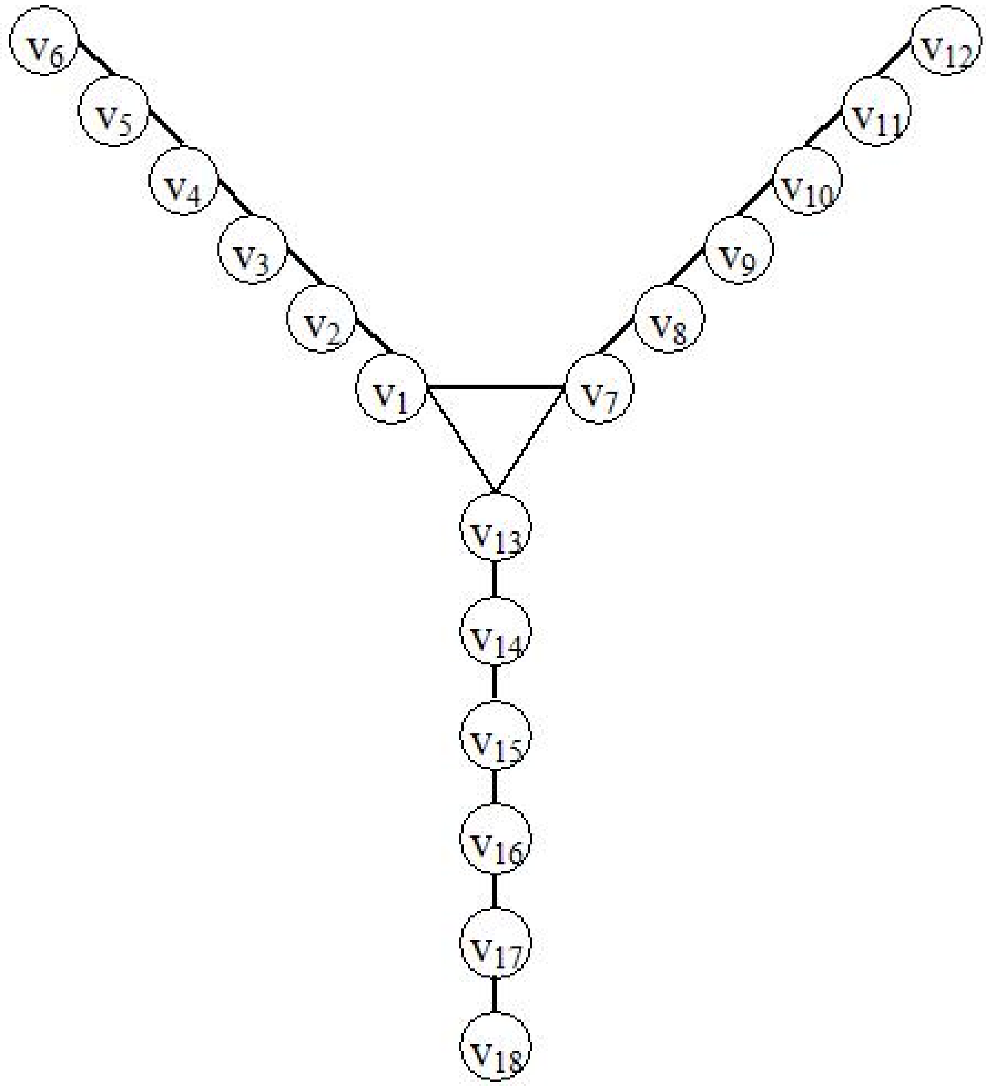


Figure 4

Consider the edge (v_1, v_7). Assume without loss of generality (as we can do because of the symmetric nature of the graph) that v_7 paid for this edge. Player v_7 could lower his cost by not buying (v_1, v_7) and instead buying (v_4, v_7). After having made this change, he would pay the same amount in edge cost as he did before, his distance to each node in $v_8, v_9, \ldots, v_18$ would be unchanged, and the sum of his distances to the nodes in $v_1, \ldots, v_6$ would be 15 instead of 21; therefore, his total cost would be lower. If the graph G in Figure 4 is $G(S)$, where $S = (s_1, \ldots, s_7, \ldots, s_8)$, and player v_7 can lower his cost by playing a different strategy that results in a different graph, then s_7 is not a best response to s_{-7}, and $G(S)$ is not stable.