ECON 425/563 // CPSC 455/555: Practice problems on information markets (Chapter 26 of AGT); answers in blue

1. Consider Example 26.8 in AGT. Explain why, after the round-1 clearing price of \$0.75 is announced, both agents know that the OR of x_1 and x_2 is 1 and thus that the process converges to an equilibrium price of 1 after one more round.

   Because he observes $x_2 = 1$, agent 2 knows from the beginning that $f(x) = 1$. Because he bid \$0.50 in round 1, and the formula for the clearing price reduces in this case to $(b_1+b_2)/2$, agent 1 knows once he sees the clearing price that agent 2 must have bid \$1.00 in round 1 and thus that agent 2 must have known that $f(x) = 1$. Therefore, agent 1 can conclude that $x_2 = 1$ and will himself bid \$1.00 in round 2, as will agent 2. The clearing price in round 2 will thus be \$1.00, and the process will terminate with an equilibrium price of \$1.00.

   According to Theorem 26.12, the assumption in this example that the common prior distribution is uniform is unnecessary. Convince yourself that this is true by working through the same example with the following (nonuniform) common prior distribution: $P(x_1 = 0) = .25$, $P(x_2 = 0) = .75$, and the two inputs are statistically independent (*i.e.*, the joint distribution on (x_1, x_2) is just the product of the distributions on x_1 and x_2). Once again, assume that agent 1 observes $x_1 = 0$ and that agent 2 observes $x_2 = 1$.

   In round 1, agent 1 bids \$0.25; this is his expected value, because the OR of the two input bits will be 1 if and only if $x_2 = 1$, which is the case with probability .25 according to the common prior. Similarly, agent 2 bids \$1.00 in round 1; because he observes $x_2 = 1$, he knows with certainty that the OR of the two input bits is 1. The clearing price in round 1 will be \$0.675. Using exactly the same reasoning process as we went through for the uniform prior, we see that agent 1 can conclude from this clearing price that $x_2 =1$ and thus that both agents will bid \$1.00 in round 2.

2. Prove that the n-input AND function satisfies the hypothesis of Theorem 26.12, *i.e.*, that the Boolean function $f(\boldsymbol{x})$ that is 1 if and only if all x_i are 1 is a weighted threshold function.

   In Definition 26.11, let $\omega_0 = 0$ and $\omega_i = 1/n$, for $1 \leq i \leq n$. Then the AND of $x_1, \ldots, x_n$ is 1 if and only if $\omega_0 + \sum_{1 \leq i \leq n} \omega_i x_i \geq 1$.

   Work through the analog of Example 26.8 for the AND function. That is, assume that there are two agents, that the common prior distribution on (x_1, x_2) is uniform, that agent 1 observes $x_1 = 0$, and that agent 2 observes $x_2 = 1$. What does each agent bid in each round, and what is the clearing price in each round?

In round 1, agent 1 bids \$0.00, because his observation that $x_1 = 0$ allows him to conclude with certainty that the AND is 0. Agent 2 bids \$0.50 in round 1, because he concludes from his observation that $x_2 = 1$ and the uniform common prior that the AND of the two input bits is 1 with probability ½. The clearing price in round 1 is therefore \$0.25. From this, agent 2 concludes that agent 1 bid \$0.00 in round 1 and thus that agent 1 must know that the AND is 0. Hence, both agents bid \$0.00 in round 2, the clearing price is \$0.00, and the process terminates with an equilibrium price of \$0.00.

3. As seen in Example 26.10, the XOR function and the uniform common prior satisfy the conditions of Theorem 26.13. Give another example of a boolean function *f* that is not expressible as a weighted threshold function and a common prior for which the price of the security *F* does not converge to the value $f(\mathbf{x})$.

   The simplest example is the negation of XOR, *i.e.*, the function $f(x)$ that is 1 if and only if the number of 1 bits in x is even, and the uniform common prior. If $n = 2$, this is the equality function EQ, i.e., the function that's 1 exactly when both input bits are 0 or both are 1. In round 1, all agents will bid \$0.50, no matter what their observations are, all will have identical posterior beliefs, and the process will converge. However, the equilibrium price of \$0.50 will not be equal to $f(x)$.

   By contrast, give an example of a common prior distribution for which the price of the security *F does* converge to the value of the XOR function.

   There are trivial examples of convergence in which the common prior distribution puts all of the probability on one pair of input bits. For a simple example with full support, consider the common prior in which $P(x_1 = 0) = P(x_2 = 0) = .10$, and the two inputs are statistically independent (i.e., the joint distribution on (x_1, x_2) is just the product of the distributions on x_1 and x_2). If agent 1 observes $x_1 = 0$ and agent 2 also observes $x_2 = 0$, then both will bid \$0.90 (= .1(0)+.9(1))[1] in round one, and the clearing price will be \$0.90; both will then conclude that the other's input is 0 and bid \$0.00 in round 2. If both input bits are 1, then the analogous calculations show that both agents will bid \$0.10 in round 1, conclude from the clearing price of \$0.10 that the other's input is 1, and bid \$0.00 in round 2. If agent 1 observes $x_1 = 0$, and agent 2 observes $x_2 = 1$, then agent 1 will bid \$0.90 (= .1(0)+.9(1)) in round 1, and agent 2 will bid \$0.10 (=.9(0)+.1(1)) in round 1. The clearing price in round 1 will be \$0.50, both agents will be able to deduce the other's input bit, and both will bid \$1.00 in round 2. A symmetric argument works for the case of $x_1 = 1, x_2 = 0$.

4. In Section 26.2 of AGT, Pennock and Sami give several reasons that *no-trade theorems* do not necessarily model real-world traders' behavior, namely the

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[1] That is, with probability .1, the input bit x_2 is 0, and the XOR is 0, and, with probability .9, the input bit x_2 is 1, and the XOR is 1.

dependence of these theorems on the assumptions of risk neutrality and common knowledge that all traders are completely rational Bayesians. Roughly speaking, these are "economic" explanations of why trade occurs despite these theorems. Give a "computational" explanation of the same phenomenon.

The no-trade theorems show that, in a fully revealing rational-expectations equilibrium, all agents have the same posterior beliefs, and thus none has an incentive to trade. However, there may not be a computationally efficient price-formation process that is guaranteed to reach such an equilibrium.