5. **Coping with NP-hardness.** Recall that the following languages are NP-complete.

- **CLIQUE** = \{(G, k), where G = (V, E) is an undirected graph, k is a positive integer, and there is a subset V’ of V such that |V’| ≥ k and (u, v) ∈ E for all pairs u, v of distinct nodes in V’\}.
- **HC** = \{G, where G = (V, E) is an undirected graph, |V| = n, and there is an ordering <v_1, ..., v_n> of the nodes in V such that (v_n, v_1) ∈ E and (v_i, v_{i+1}) ∈ E, for 1 ≤ i ≤ n-1\}.
- **COLORABLE** = \{(G, k), where G = (V, E) is an undirected graph, k is a positive integer, and there is a function f: V → \{1, 2, ..., k\} such that f(u) ≠ f(v) if (u, v) ∈ E\}.

Prove that the following special cases of these language are in P.

(a) (2 points) **k_0-CLIQUE** = \{G, where G = (V, E) is an undirected graph, and there is a subset V’ of V such that |V’| ≥ k_0 and (u, v) ∈ E for all pairs u, v of distinct nodes in V’\}. (Here, k_0 is a fixed, positive integer, i.e., one that does not depend on the size of the input graph.)

(b) (4 points) **DEGREE-2-HC** = \{G, where G = (V, E) is an undirected graph; |V| = n; for each u ∈ V, there are at most two other nodes v and w such that (u, v) ∈ E and (u, w) ∈ E; and there is an ordering <v_1, ..., v_n> of the nodes in V such that (v_n, v_1) ∈ E and (v_i, v_{i+1}) ∈ E, for 1 ≤ i ≤ n-1\}

(c) (6 points) **2-COLORABLE** = \{G, where G = (V, E) is an undirected graph, and there is a function f: V → \{1, 2\} such that f(u) ≠ f(v) if (u, v) ∈ E\}

6. **Basic complexity classes.** Prove that P ⊆ NP ⊆ PSPACE.

7. **A game on directed graphs.** The language GEOGRAPHY is defined as follows. An instance consists of a directed graph G = (V, A) and a designated start node s ∈ V. Player I moves first by choosing node s; then player II moves by choosing a node s’ ≠ s such that (s, s’) ∈ A. More generally, after m moves have been made, exactly m nodes have been chosen, and one of the two players has chosen node u in the m_th move; the (m+1)_th move is then made by the other player, who must choose a node v such that (u, v) ∈ A, and v has not already been chosen in one of the first m moves. When a player is unable to move (because no such node v exists), he loses. The instance (G, s) is a yes-instance of GEOGRAPHY if and only if player I has a winning strategy.

(a) (1 points) Construct a yes-instance of GEOGRAPHY.

(b) (1 point) Construct a no-instance of GEOGRAPHY.

(c) (10 points) Prove that GEOGRAPHY is in PSPACE.
8. **Computing equilibria in games**: Give an algorithm that takes as input a two-player game in normal form and produces as output a Nash Equilibrium of the game. (You should use the definitions of “game in normal form” and “Nash Equilibrium” that were given in Lecture I on September 9, 2008.) You need not give a polynomial-time algorithm, but you must explain what the size $n$ of a problem instance is and give upper bounds on the time complexity $T(n)$ and the space complexity $S(n)$ of your algorithm. It may be useful first to describe an algorithm that finds a pure-strategy Nash equilibrium if one exists and then modify it to accommodate the possibility that a mixed-strategy Nash equilibrium is needed.