Economics and Computation

ECON 425/563 and CPSC 455/555

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Lecture I

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1 Outline

The review of basic microeconomic theory will be organized as follows:

1. Games with Complete Information
2. Games with Incomplete Information
3. Mechanism Design

The first two parts take the structure of the **game** as given and characterize **solutions**. In this context, "a game is a description of strategic interaction that includes the constraints on the actions that the players *can* take and the players' interests, but does not specify the actions that the players *do* take. A solution is a systematic description of the outcomes that may emerge in a family of games."[1] Part 2. distinguishes itself from part 1. in that it analyzes games where some players are uncertain about the payoffs (own or others), the strategies or the players in the game.
The last part 3. reverses the logic of the previous two parts. Now, a planner has a certain outcome to be realized in his mind and wants to create a game to be played by the agents such that the particular outcome is materialized. Moving away from a centralized, planned situation, the question is asked how a desired outcome can still be achieved in a decentralized environment in which agents pursue their self-interest, e.g. on the Internet.

Historically, game theory takes off with **John von Neumann** and his article "On the Theory of Parlor Games" from 1928. These ideas were incorporated in the book "Games and Economic Behavior" that he has written together with **Oskar Morgenstern** in 1944. Their ideas were significantly extended by **John Nash** in 1950 and 1951. After the work of these founding fathers, a large body of research has developed, that has achieved its latest climax with Eric Maskin, Leo Hurwicz and Roger Myerson being awarded the Nobel Prize in Economics in the year 2007 for the development of mechanism design.

1.1 Literature

The following two textbooks contain an accessible and detailed description of the material covered in this introduction to microeconomic theory:

- Martin Osborne and Ariel Rubinstein: "A Course in Game Theory", [OR94].
- Martin Osborne: "An introduction to Game Theory", [Osb04].

[1]See page 2 of the textbook by Martin Osborne and Ariel Rubinstein.

2 Games with Complete Information

2.1 Components

A game with complete information consists of the following components:

- A set of players $\mathcal{I} = \{1, ..., I\}$, where one player is denoted by $i \in \mathcal{I}$.
- A set of actions A_i for each player $i \in \mathcal{I}$, where $A_i = \{a_i^1, ..., a_i^K\}$. One particular action for player i is denoted by $a_i \in A_i$.
- A payoff function u_i for each player $i \in \mathcal{I}$ that maps a tuple of actions, one by each player, into the real numbers, i.e.

$$u_i : A_1 \times ... \times A_I \to \mathbb{R}.$$

If one wants to refer to the possible tuples of actions that can be taken by the entity of players, one writes

$$A := A_1 \times ...A_I.$$

A typical element from A is denoted by a. Under certain circumstances it makes sense to distinguish between actions taken by a particular player i and all other players involved in the game, denoted by $-i$. Notationally, one refers to an action taken by any player but player i as

$$a_{-i} := \{a_1, ..., a_{i-1}, a_{i+1}, ..., a_I\}.$$

So, a typical element $a \in A$ can be written as

$$a = (a_i, a_{-i})\text{for any } i \in \mathcal{I}.$$

John von Neumann analyzed so-called zero-sum games, i.e. games with $I = 2$ (two-player games) which satisfy the condition

$$u_1(a) + u_2(a) = 0.$$

These are situations in which any player's gain or loss is exactly offset by the other player's loss or gain. It was John Nash who extended the setting under consideration to $I > 2$ and to non-zero-sum games. In a sense, John von Neumann's thinking corresponds to the political circumstances - the Cold War - that were present after the publication of his book with Oskar Morgenstern. Involving two opposing parties, the analogy to zero-sum games is almost immediate. John Nash's extension corresponds much more to modern economic thinking. For example, trading activity between two countries involve gains that do not fit the description of a zero-sum game, with the possibility that the gains from trade are bigger for one party than for another. As another example, the application of these game-theoretic settings to computer science involve $I >> 2$.

Definition 1 *A **game in normal form** Γ is given by*

$$\Gamma = \left\{ I, \{A_i\}_{i=1}^{I}, \{u_i\}_{i=1}^{I} \right\}.$$

Remark 1 *For non-economists the use of a utility-function in the definition of a game in normal form might appear alienating. In fact, this concept can be underpinned choice-theoretically. One can start with a binary preference relation $\succeq$. In this context, $x \succeq y$ means that a person likes x (weakly) more than y. It was John von Neumann who has shown that, if a preference relation on a finite set of choices satisfies*

- *completeness*
  (From the underlying set that contains all possible choices, any two possible elements can be "sorted" by $\succeq$.)
  and
- *transitivity,*
  (For any three elements x, y, z from the underlying set of choices satisfying $x \succeq y$ as well as $y \succeq z$, it follows that $x \succeq z$.)

then the player's preference-relation can be represented by a utility function.
There is a multitude of extensions to this result, involving non-finite sets over which the preference-relation is defined etc.

Remark 2 *An important point to be made is the fact that the structure of the game is always common knowledge among the players, i.e. known to anyone involved in the game.*

2.2 Solution-Concepts

Subsequent to the structure of the games under consideration, the focus will be shifted towards reasonable predictions that can be made about the outcome of a particular game. Different solution-concepts and their usefulness will be outlined along the lines of different examples.

2.2.1 Dominance-Solvability

Consider the following famous game, the so-called **Prisoner's Dilemma**[2]:

[2] Caveat: The analogous game in the textbook "Algorithmic Game Theory", [NRTV08], is formulated in terms of costs. In order to obtain utility, one needs to multiply the costs by -1.

Column-Player

		Silent	Confess
Row-Player	Silent	3,3	0,4
	Confess	4,0	1,1

This so-called payoff-matrix must be read as follows:
There are only two players involved in this game, a row-player and a column-player. Each of these players has two choices available, "Silent" or "Confess". For any pair of actions that can be taken by the two players, the corresponding field of the payoff-matrix shows the pair of playoffs for the players. By convention, the row-player's payoff is listed first and the column-player's payoff is listed second.

The following story can be used to motivate this game:
The two players are accused of conspiring in two crimes, one minor crime for which their guilt can be proved without any confession, and one major crime for which they can be convicted only if at least one confesses. The prosecutor promises that, if exactly one confesses, the confessor will be go free now (utility 4) but the other will get a severe sentence (utility 0). If both confess, then they will get a sentence that is only slightly less severe (utility 1). If neither confesses then they both get a light sentence for the minor crime (utility 3).[3]

If one looks at the situation of the row-player, one can observe that he always receives a strictly higher payoff from "confess" than from "silent". If the column-player plays "Silent", the row-player receives 4 from "Confess", but only 3 from "Silent". If the column-player plays "Confess", the row-player receives 1 from "Confess", but only 0 from "Silent". The situation for the column-player is completely analogous because the game is symmetric.
This argument motivates the following definition:

Definition 2 *An action $a_i \in A_i$ is a **dominant strategy** for player i iff*

$$u_i(a_i, a_{-i}) > u_i(a'_i, a_{-i}) \qquad \forall a'_i \neq a_i, \forall a_{-i}. \tag{1}$$

Hence, an action $a_i \in A_i$ is a dominant strategy if, irrespective of the other players' actions ($\forall a_{-i}$), it yields a strictly higher payoff than any other action a'_i for player i.

[3]The previous description of the *Prisoner's dilemma* is taken from the textbook by Roger Myerson, [Mye97].

Clearly, in the *Prisoner's dilemma* the action "Confess" is a strictly dominant strategy for both players. Hence, the outcome ("Confess","Confess") is called an equilibrium in dominant strategies.

Sometimes, the strict inequality in (1) is too strict as a condition to the payoff-structure. The following definition comprises a slightly weaker notion of dominance:

Definition 3 *An action $a_i \in A_i$ is a* ***weakly dominant strategy*** *for player i iff*

$$u_i(a_i, a_{-i}) \geqslant u_i(a'_i, a_{-i}) \qquad \forall a'_i \neq a_i, \forall a_{-i}. \tag{2}$$

and $\exists a'_{-i}$ such that

$$u_i(a_i, a'_{-i}) > u_i(a'_i, a'_{-i}) \qquad \forall a'_i \neq a_i. \tag{3}$$

This definition allows for equality of certain payoffs. As long as there is at least one action by the other players for which the action a_i is strictly better than all other actions available to player i, the notion of dominance can still be applied.

2.3 Best Responses

Now consider the following game, the so-called **Battle of the Sexes**:

		Sheila Opera	**Sheila** Football
Bruce	Opera	1,2	0,0
Bruce	Football	0,0	2,1

The following story can be told about the *Battle of the Sexes*:
Imagine a couple, Bruce and Sheila, who are making plans for the weekend. They can either go to the opera or attend a football-game. Bruce's favorite is the football-game, but Sheila's favorite is the opera. Nevertheless, both prefer being together with their partner over attending one of the events on their own. The decision where to go has to be made simultaneously and is irreversible.

This game exhibits the following characteristics:

- This is an interactive component to this game since there is no dominant strategy for each of the players:

- For Bruce, O is NOT a better choice than F irrespective of what Sheila is doing. If she chooses F, O gets Bruce a strictly worse payoff than F.
- On the other side, for Bruce, F is NOT a better choice than B either. If she chooses O, then Bruce receives a strictly lower payoff from F than he would get from O.
- The same argument applies for Sheila. Due to the symmetry of the game, you simply have to exchange the name-labels "Bruce" and "Sheila" in the previous two arguments.

So, it will depend on the other player's action which action gives a player the highest payoff.

- There is an element of conflict in that each player obtains an extra-surplus for his own favorite choice.
- There is also an element of coordination in that both players receive a payoff of 0 if the players' choices do not agree.
- There is no natural way to make an assumption about the other person's action before the decisions need to be submitted.
- Comparing the *Prisoner's dilemma* to the situation of the *Battle of the Sexes*, one obtains:
  If the game has ended up at an unfavorable outcome, i.e. an outcome where both players can actually be better off at another outcome.[4] this corresponds to the outcome $(1, 1)$ in the *Prisoner's dilemma* and $(0, 0)$ in the *Battle of the Sexes*. In the *Prisoner's dilemma*, the players would have to coordinate themselves and deviate jointly in order to achieve the outcome $(3, 3)$, whereas in the *Battle of the Sexes* unilateral deviations are sufficient to move the outcome to either $(1, 2)$ or $(2, 1)$.

So, the solution-concept that will be applied to characterize the desired outcomes of this game is the concept of a **Nash-equilibrium**. But in order to formally define this concept, one first needs to characterize a player's best response to another players's strategy:

Definition 4 *a_i is a **best response to action profile** a_{-i} iff*

$$u_i(a_i, a_{-i}) \geqslant u_i(a'_i, a_{-i}), \qquad \forall a'_i \in A_i. \tag{4}$$

In words, one fixes one particular action of the other players a_{-i} and looks for the best action given a_{-i}. In contrast to a dominant strategy, being a best response is only a local

[4] This is the notion of Pareto-inferiority.

property of an action.
From the definition of a best response, one obtains the notion of a **Nash-equilibrium** as follows:

Definition 5 $a^* = (a_1^*, a_2^*, ..., a_I^*)$ *is a* ***(pure strategy) Nash-equilibrium*** *iff for all players* $i \in \{1, ..., I\}$

$$u(a_i^*, a_{-i}^*) \geqslant u(a_i, a_{-i}^*), \qquad \forall a_i \in A_i. \tag{5}$$

Inherent to the definition of a Nash-equilibrium is a notion of stability. None of the players wants to choose another action than the action of the Nash-equilibrium because no player can make himself strictly better off by choosing another action. Put differently, a Nash-equilibrium is a collection of mutual best responses for all players of the game. The best-response correspondence $BR_i(a_{-i})$ of player $i \in \mathcal{I}$ is defined by

$$a_i \in BR_i(a_{-i}) :\Leftrightarrow u_i(a_i, a_{-i}) \geqslant u_i(a_i', a_{-i}).$$

So, this correspondence simply picks out the best response for player i given the other players' action-profile a_{-i}. All players' best-responses correspondences are stacked into the correspondence BR that is defined as follows:

$$BR(a) = (BR_1(a_{-1}), ..., BR_I(a_{-I})).$$

Now, the definition of the action-profile a^* being a Nash-equilibrium corresponds to a^* being a fixed point of the best-response correspondence BR:

$$a^* \in BR(a^*).$$

The fact, that a tuple of dominant strategies for each player is a Nash-equilibrium, is evident from the definition of dominant strategies and that of Nash-equilibria. So, the outcome ("Confess","Confess") is also a Nash-equilibrium of the *Prisoner's dilemma*. Hence, it makes sense to scan for dominant strategies before one attempts to find Nash-equilibria.

There are two possible justifications for the notion of a Nash-equilibrium as a solution-concept:

- A Nash equilibrium is a steady state if players are subject to certain social norms and conventions.
- A Nash equilibrium is the result of a mutual inductive process of reasoning.

In practice, Nash-equilibria (in pure strategies) can be computed as follows:

One needs to determine the outcomes that are mutual best responses. One can start with any action that any player can take. So, for example, fix Sheila's choice "Opera". Now, it is Bruce's best response to Sheila's choice O to choose O. This choice will be marked with a little dash:

		Sheila	
		Opera	Football
Bruce	Opera	$\underline{1}$,2	0,0
	Football	0,0	2,1

Now, fix Sheila's choice "Football". It is a best response for Bruce to Sheila's choice F to play F. Mark it with another dash:

		Sheila	
		Opera	Football
Bruce	Opera	$\underline{1}$,2	0,0
	Football	0,0	$\underline{2}$,1

Now, we are done with all of Sheila's actions. So, one switches to Bruce and fixes, for example, his choice O. For Sheila, it is a best response to Bruce's choice O to choose O. So, put another dash below Sheila's payoff from O in the field (O, O):

		Sheila	
		Opera	Football
Bruce	Opera	$\underline{1}$,$\underline{2}$	0,0
	Football	0,0	$\underline{2}$,1

Fixing Bruce's action F, it is a best response for Sheila to this action to choose F, too.

This completes the analysis of the player's best responses:

Sheila

Bruce		Opera	Football
	Opera	$\underline{1}$,$\underline{2}$	0,0
	Football	0,0	$\underline{2}$,$\underline{1}$

Now, you have found a Nash-equilibrium if there is a payoff-combination where both numbers have dashes. In other words, such a field is a strategy-pair that consists of mutual best responses.

Hence, in the *Battle of the Sexes*, the set of (pure-strategy) Nash-equilibria is given by

$$\{(0,0),(F,F)\}.$$

For each of these pairs of outcomes, none of the players has an incentive to choose any other action given the other player's action. Put differently, no player has a profitable deviation.

Remark 3 *The following elementary procedure only works for games in normal form, i.e. games that can be represented by a payoff-matrix. Furthermore, it will only find pure-strategy Nash-equilibria.*[5] *But, it will find all pure-strategy Nash-equilibria, i.e. a pair of actions is NOT a pure-strategy Nash-equilibrium, if (at least) one payoff does not have a dash.*

As another example for the solution-concept of Nash-equilibrium consider the so-called game of **Hawk vs. Dove**, which is sometimes also referred to as the **Game of Chicken**:

Player 2

Player 1		Hawk	Dove
	Hawk	0,0	7,2
	Dove	2,7	6,6

[5] Mixed-strategies will be introduced below.

With respect to the name *Game of Chicken* the following story in rememberance of James Dean can be told:
Imagine two car-drivers that drive fast towards each other on a narrow road. At the point where they will meet each other, there are small parking booths on either side of the road. If one of the drivers decides to drive into the booth, then the car has to be stopped completely until the other car has passed. If none of the cars stops, then there will be an accident on the narrow road (both drivers receive utility 0). If one car stops, then the driver of the car that can continue to drive fast on the road feels enthusiastic (utility 7), whereas the driver of the stopped car is glad not to have had an accident (utility 2). If both cars stop at their booths, the drivers gently smile at each other and drive slowly past each other (both receive utility 6).

Applying the previously outlined procedure will yield the (pure-strategy) Nash-equilibria of this game:

Start, for example, with player 2's choice of "Hawk". Then, it is a best response for player 1 to this action to choose "Dove":

Player 2

		Hawk	Dove
Player 1	Hawk	0,0	$\underline{7}$,2
	Dove	2,7	6,6

Fixing the action "Dove" for player 2, it is a best response for player 1 to player 2's action "Dove" to choose "Hawk":

Player 2

		Hawk	Dove
Player 1	Hawk	0,0	$\underline{7}$,2
	Dove	$\underline{2}$,7	6,6

Now, fix player 1's action "Hawk". Then, it is a best response for player 2 to this action to choose "Dove":

Player 2

Player 1		Hawk	Dove
	Hawk	0,0	7,2
	Dove	2,7	6,6

Fixing player 1's action "Dove", it is a best response for player 2 to this action to choose "Hawk":

Player 2

Player 1		Hawk	Dove
	Hawk	0,0	7,2
	Dove	2,7	6,6

So, the set of (pure-strategy) Nash-equilibria of *Hawk vs. Dove* is given by

$$\{(H, D), (D, H)\}.$$

In other words, none of the players has a profitable deviation from one of the two (pure-strategy) Nash-equilibria.

2.4 Mixed Strategies

There are games which do not have pure-strategy Nash equilibria. As an example, consider the following game, which is called **Matching Pennies**:

Player 2

Player 1		Heads	Tails
	Heads	1,-1	-1,1
	Tails	-1,1	1,-1

The following story can be told for this game:
Imagine player 1 and player 2 are both holding a penny in their hand. Each of the coins has a head-side and a tail-side. On a specific command, both players simultaneously have to open their hand so that one side of their penny shows upward. If the two sides of the pennies match, then player 1 receives player 2's penny. If they do not match, then player 2 receives player 1's penny.

Remarkably, this game is a zero-sum game, as it has been analyzed by John von Neumann. This can be seen from the fact that the sum of the players' payoffs in each matrix-cell is zero.

This game does not have any Nash-equilibria in pure-strategies. Most easily, this can be seen from the previously outlined procedure:

		Player 2	
		Heads	Tails
Player 1	Heads	$\underline{1}$,-1	-1,$\underline{1}$
	Tails	-1,$\underline{1}$	$\underline{1}$,-1

None of the cells of the matrix bears a pair of dashes below the pair of payoffs, implying the non-existence of pure-strategy Nash-equilibria.

In view of the previous result, one would like to generalize the notion of a strategy to involve randomization over different actions. Formally, this yields the notion of **mixed strategies** defined as follows:

Definition 6 *Let a player's set of pure actions be given by $A_i = \{a_i^1, ..., a_i^k\}$. Then, a* ***mixed strategy*** σ_i *is defined as*

$$\sigma_i : A_i \rightarrow [0, 1],$$

such that

$$\sum_{a_i \in A_i} \sigma_i(a_i) = \sum_{l=1}^{k} \sigma_i(a_i^k) = 1.$$

So, by playing a mixed strategy, a player $i \in \mathcal{I}$ does not choose a single action but chooses a probability distribution over a certain amount of actions. Obviously, the notion of a pure strategy is incorporated into the definition of mixed strategies by putting all the probability mass on one single action.

Remark 4 *Every time that mixed strategies will show up in the following, it will be assumed that all players compute their utility according to the Expected-Utility specification.*

Concerning notation, the $-i$-notation for the strategy of all players but player i trivially carries over to mixed strategies. The notion of a best response carries over to mixed strategies if one simply replaces a_i by σ_i in (4). Furthermore, the definition of a Nash-equilibrium can be extended to capture mixed strategies by replacing $a^* = (a_i^*, a_{-i}^*)$ by $\sigma^* = (\sigma_i^*, \sigma*_{-i})$ in (5).
The fact that the probability-simplex over any finite set of actions is always a compact set may raise questions about the computability of Nash-equilibria (optimization over compact sets) or the existence of Nash-equilibria for general games (fix-point arguments). Concerning the computability, it was John von Neumann who, without defining Nash-equilibria explicitly, has given a characterization of Nash-equilibria via linear programming in the context of zero-sum games. Concerning the existence of Nash-equilibria, John Nash has used the compactness of the probability-simplex to apply Kakutani's fix-point theorem, an elaborate version of Brouwer's fix-point theorem, to demonstrate the existence of Nash-equilibria for fairly general finite games.

Now, we will come to the question of how to actually compute mixed-strategy Nash-equilibria. A property that is fundamentally important for this computation is the following:

Any player must be indifferent between all the actions that receive positive weight in the player's mixed strategy.

This can be seen as follows:
Suppose σ^* is a Nash-equilibrium. Fix the other players' strategy to be σ_{-i}. If σ_i involves player i putting positive weight on two actions a_i^1 and a_i^2, then it is impossible that, for σ_{-1}, a_i^1 yields a strictly higher payoff than a_i^2 or vice versa. If this were the case, then it would never be a best response to mix over the two actions, but it would be strictly better to choose the "preferred" action for sure.

Now, the procedure to determine mixed strategy Nash-equilibria will be outlines in the context of the *Matching Pennies* game:

Start with any player, for example player 1. How can this player be made indifferent between H and T? Suppose that player 2 puts probability-weight $\beta \in [0, 1]$ on H (hence, weight $1 - \beta$ on T). Then, player 1's payoffs are:

$$\begin{aligned} \text{for H} &: \beta \cdot 1 + (1 - \beta) \cdot (-1), \\ \text{for T} &: \beta \cdot -1 + (1 - \beta) \cdot 1. \end{aligned}$$

Indifference between H and T therefore requires:

$$\begin{aligned} \beta \cdot 1 + (1-\beta) \cdot (-1) &= \beta \cdot (-1) + (1-\beta) \cdot 1, \\ \beta &= \frac{1}{2}. \end{aligned}$$

So, for a mixed-strategy Nash-equilibrium one requires player 2 to play $\frac{1}{2}H + \frac{1}{2}T$.
Now, consider player 2. How can this player be made indifferent between H and T? Suppose that player 1 puts probability-weight $\alpha \in [0, 1]$ on H (hence, weight $1 - \alpha$ on T). Then, player 2's payoffs are:

$$\begin{aligned} \text{for H} &: \alpha \cdot (-1) + (1-\alpha) \cdot 1, \\ \text{for T} &: \alpha \cdot 1 + (1-\alpha) \cdot (-1). \end{aligned}$$

Indifference between H and T therefore requires:

$$\begin{aligned} & \alpha \cdot (-1) + (1-\alpha) \cdot 1 = \alpha \cdot 1 + (1-\alpha) \cdot (-1) \\ \Leftrightarrow \quad & \alpha = \frac{1}{2} \end{aligned}$$

So, for a mixed-strategy Nash-equilibrium one requires player 1 to play $\frac{1}{2}H + \frac{1}{2}T$. In summary, the Nash-equilibrium (there is in fact only this one) for *Matching Pennies* is given by

$$\left(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T\right).$$

Coming back to the game *Battle of the Sexes*, one can find another, a mixed-strategy Nash-equilibrium of this game via the following graphical argument:
Assume that Sheila puts probability weight σ_S on "Opera", i.e.

$$\sigma_S := \sigma_S(O).$$

Then, Bruce's payoff is:

$$\begin{aligned} \text{for O} &: \sigma_S \cdot 1 + (1-\sigma_S) \cdot 0, \\ \text{for T} &: \sigma_S \cdot 0 + (1-\sigma_S) \cdot 2. \end{aligned}$$

In order for Bruce to be indifferent between "Opera" and "Football", one therefore needs

$$\begin{aligned} & \sigma_S \cdot 1 + (1-\sigma_S) \cdot 0 = \sigma_S \cdot 0 + (1-\sigma_S) \cdot 2 \\ \Leftrightarrow \quad & \sigma_S = \frac{2}{3}. \end{aligned}$$

Hence, for $\sigma_S < \frac{2}{3}$, Bruce obtains a strictly higher payoff from "Football" than from "Opera". In contrast, for $\sigma_S > \frac{2}{3}$, Bruce obtains a strictly higher payoff from "Opera" than from "Football". Denoting by σ_B the probability weight that Bruce puts on "Opera", i.e.

$$\sigma_S := \sigma_S(O),$$

one can formulate Bruce's best-response correspondence σ_B^* as follows:

$$\sigma_B^*(\sigma_S) = \begin{cases} 0 & \text{for } \sigma_S < \frac{2}{3}, \\ \lambda O + (1-\lambda)F \text{ for any } \lambda \in [0,1] & \text{if } \sigma_S = \frac{2}{3}, \\ 1 & \text{for } \sigma_S > \frac{2}{3} \end{cases} .$$

Now, Sheila's payoff for any probability σ_B that Bruce puts on "Opera" is:

$$\begin{aligned} \text{for O} &: \quad \sigma_B \cdot 2 + (1-\sigma_B)\cdot 0, \\ \text{for T} &: \quad \sigma_B \cdot 0 + (1-\sigma_B)\cdot 1. \end{aligned}$$

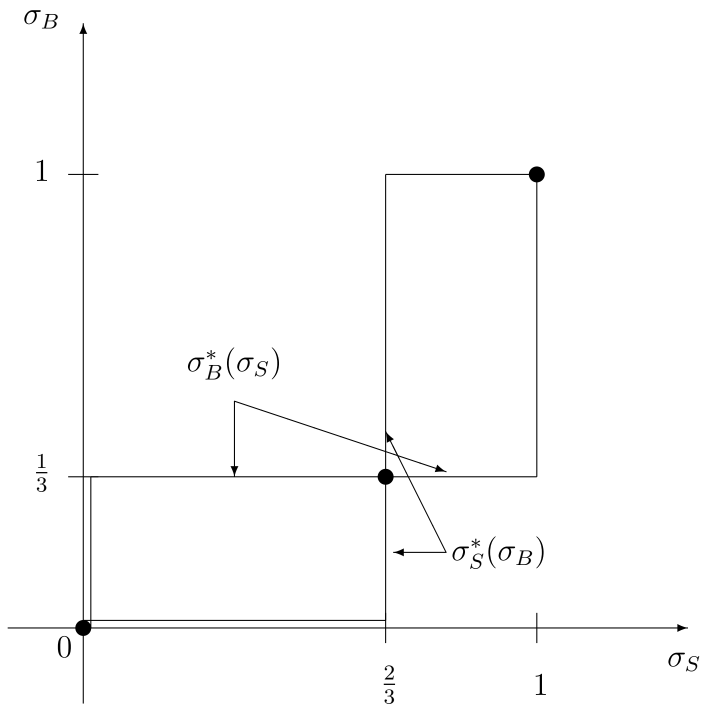
In order for Sheila to be indifferent between "Opera" and "Football", one therefore needs

$$\begin{aligned} & \sigma_B \cdot 2 + (1-\sigma_B)\cdot 0 = \sigma_B \cdot 0 + (1-\sigma_B)\cdot 1 \\ \Leftrightarrow \quad & \sigma_S = \frac{1}{3}. \end{aligned}$$

Hence, for $\sigma_B < \frac{1}{3}$, Sheila obtains a strictly higher payoff from "Football" than from "Opera". In contrast, for $\sigma_S > \frac{1}{3}$, Sheila obtains a strictly higher payoff from "Opera" than from "Football". Hence Sheila's best-response correspondence σ_S^* is given by:

$$\sigma_S^*(\sigma_B) = \begin{cases} 0 & \text{for } \sigma_S < \frac{1}{3}, \\ \lambda O + (1-\lambda)F \text{ for any } \lambda \in [0,1] & \text{if } \sigma_S = \frac{1}{3}, \\ 1 & \text{for } \sigma_S > \frac{1}{3} \end{cases} .$$

Depicting both best-response correspondences graphically, one obtains the three Nash-equilibria of the game (2 in pure strategies, 1 in mixed strategies) as he intersections of the best-response correspondences (i.e. as point that are mutually best responses):



Summing up, the set of Nash-equilibria for the game *Battle of the Sexes* is given by

$$\left\{(O,O),(F,F),\left(\frac{1}{3}O+\frac{2}{3}F,\frac{2}{3}O+\frac{1}{3}F\right)\right\}$$

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