Linear Congruence Equations

Let \( a, x \in \mathbb{Z}_n^* \). Recall that \( x \) is said to be an inverse of \( a \) modulo \( n \) if \( ax \equiv 1 \) (mod \( n \)). It is easily seen that the inverse, if it exists, is unique modulo \( n \), for if \( ax \equiv 1 \) (mod \( n \)) and \( ay \equiv 1 \) (mod \( n \)), then \( x \equiv xay \equiv y \) (mod \( n \)). We denote this unique \( x \), when it exists, by \( a^{-1} \) (mod \( n \)) (or simply \( a^{-1} \) when the modulus \( n \) is clear from context).

**Theorem 1**

Let \( a \in \mathbb{Z}_n^* \). Then \( a^{-1} \) exists in \( \mathbb{Z}_n^* \).

**Proof:** Let \( a \in \mathbb{Z}_n^* \) and consider the function \( f_a(x) = ax \mod n \). \( f_a \) is easily shown to be a one-one mapping from \( \mathbb{Z}_n^* \) to \( \mathbb{Z}_n^* \). Hence, \( f_a \) is also onto, so for some \( x \in \mathbb{Z}_n^* \), \( f_a(x) = 1 \). Then \( ax \equiv 1 \) (mod \( n \)), so \( x = a^{-1} \) (mod \( n \)). \( \square \)

We showed in class how to use the Extended Euclidian algorithm to efficiently compute \( a^{-1} \) (mod \( n \)) given \( a \) and \( n \).

Here we consider the solvability of the more general linear congruence equation:

\[
ax \equiv b \pmod n
\]  

where \( a, b \in \mathbb{Z}_n^* \) are constants, and \( x \) is a variable ranging over \( \mathbb{Z}_n^* \).

**Theorem 2**

Let \( a, b, n \in \mathbb{Z}_n^* \). Let \( d = \gcd(a, n) \). If \( d \mid b \) then \( ax \equiv b \pmod n \) has \( d \) solutions \( x_0, \ldots, x_{d-1} \), where

\[
x_t = \left( \frac{b}{d} \right) \bar{x} + \left( \frac{n}{d} \right) t
\]  

and \( \bar{x} = (\frac{a}{d})^{-1} \pmod{\frac{n}{d}} \). If \( d \nmid n \), then \( ax \equiv b \pmod n \) has no solutions.

**Proof:** Let \( d = \gcd(a, n) \). Clearly if \( ax \equiv b \pmod n \), then \( d \mid b \), so there are no solutions if \( d \nmid b \).

Now suppose \( d \mid b \). Since \( (\frac{a}{d}) \) and \( (\frac{n}{d}) \) are relatively prime, \( \bar{x} \) exists by Theorem 1. Multiplying both sides of (2) by \( a \), we get

\[
a x_t = b \left( \frac{a}{d} \right) \bar{x} + n \left( \frac{a}{d} \right) t
\]  

where now we are working over the integers. But \( \left( \frac{a}{d} \right) \bar{x} = 1 + kn \frac{a}{d} \) for some \( k \) by the definition of \( \bar{x} \), so substituting for \( \left( \frac{a}{d} \right) \bar{x} \) in (3) yields

\[
a x_t = b + kn \left( \frac{b}{d} \right) + n \left( \frac{a}{d} \right) t
\]  

The quantities in parentheses are both integers, so it follows immediately that \( ax_t \equiv b \pmod n \) and hence \( x_t \) is a solution of (1).

It remains to show that the \( d \) solutions above are distinct modulo \( n \). But this is obvious since \( x_0 < x_1 < \ldots < x_{d-1} \) and \( x_{d-1} - x_0 = \frac{n}{d}(d - 1) < n \). \( \square \)