CPSC 467b: Cryptography and Computer Security
Lecture 19

Michael J. Fischer

Department of Computer Science
Yale University

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1. Zero Knowledge Interactive Proofs
   - Secret cave protocol
   - ZKIP for graph isomorphism
   - Abstraction from two ZKIP examples

2. Other Kinds of Interactive Proofs
   - Interactive proof of graph non-isomorphism

Zero Knowledge Interactive Proofs (ZKIP)

Zero knowlege interactive proofs (ZKIP)

A round of the simplified Feige-Fiat-Shamir protocol is an example of a so-called *zero-knowledge interactive proof*.

These are protocols where Bob provably learns nothing about Alice's secret.

Here, "learns" means computational knowledge: Anything that Bob could have computed with the help of Alice he could have computed by himself without Alice's help.

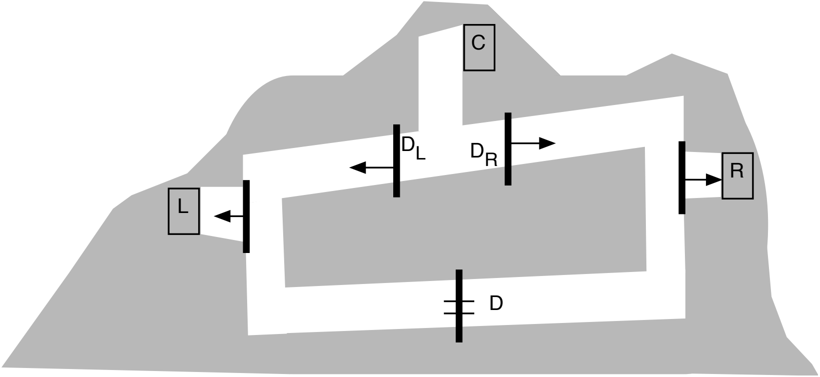
We now consider zero knowledge proofs in greater detail.

The Secret Cave Protocol

The secret cave protocol

The secret cave protocol illustrates the fundamental ideas behind zero knowledge without any reference to number theory or hardness of computation.

Image a cave with tunnels and doors as shown below.



Secret cave protocol (cont.)

There are three openings to the cave: L, C, and R.

L and R are blocked by exit doors, like at a movie theater, which can be opened from the inside but are locked from the outside. The only way into the cave is through passage C.

The cave itself consists of a U-shaped tunnel that runs between L and R. There is a locked door D in the middle of this tunnel, dividing it into a left part and a right part.

A short tunnel from C leads to a pair of doors D_L and D_R through which one can enter left and right parts of the cave, respectively. These doors are also one-way doors that allow passage from C into either the left or right parts of the cave, but once one passes through, the door locks behind and one cannot return to C.

Alice's proposition

Alice approaches Bob, tells him that she has a key that opens door D, and offers to sell it to him.

Bob would really like such a key, as he often goes into the cave to collect mushrooms and would like easy access to both sides of the cave without having to return to the surface to get into the other side.

However, he doesn't trust Alice that the key really works, and Alice doesn't trust him with her key until she gets paid.

Their conversation

Bob tells Alice.

> *"Give me the key so I can go down into the cave and try it to make sure that it really works."*

Alice retorts,

> *"I'm not that dumb. If I give you the key and you disappear into the cave, I'll probably never see either you or my key again. Pay me first and then try the key."*

Bob answers,

> *"If I do that, then you'll disappear with my money, and I'm likely to be stuck with a non-working key."*

How do they resolve their dilemma?

They think about this problem for awhile, and then Alice suggests,

> *"Here's an idea: I'll enter the cave through door C, go into the left part of the cave, open D with my key, go through it into the right part of the cave, and then come out door R. When you see me come out R, you'll know I've succeeded in opening the door."*

Bob thinks about this and then asks,

> *"How do I know you'll go into the left part of the cave? Maybe you'll just go into the right part and come out door R and never go through D."*

Alice's plan

Alice says,

> *"OK. I'll go into either the left or right side of the cave. You'll know I'm there because you'll hear door D_L or D_R clank when it closes behind me. You then yell down into the cave which door you want me to come out—L or R—and I'll do so. If I'm on the opposite side from what you request, then I'll have no choice but to unlock D in order to pass through to the other side."*

Bob's hesitation

Bob is beginning to be satisfied, but he hesitates.

> *"Well, yes, that's true, but if you're lucky and happen to be on the side I call out, then you don't have to use your key at all, and I still won't know that it works."*

Alice answers,

> *"Well, I might be lucky once, but I surely won't be lucky 20 times in a row, so I'll agree to do this 20 times. If I succeed in coming out the side you request all 20 times, do you agree to buy my key?"*

Bob agrees, and they spend the rest of the afternoon climbing in and out of the cave and shouting.

ZKIP for graph isomorphism

Graph isomorphism problem

Two undirected graphs G and H are said to be *isomorphic* if there exists a bijection π from vertices of G to vertices of H that preserves edges.

That is, $\{x, y\}$ is an edge of G iff $\{\pi(x), \pi(y)\}$ is an edge of H.

No known polynomial time algorithm decides, given two graphs G and H, whether they are isomorphic, but this problem is also not known to be *NP*-hard.

It follows that there is no known polynomial time algorithm for finding the isomorphism π given two isomorphic graphs G and H. Why?

A zero-knowledge proof for isomorphism

Now, suppose G_0 and G_1 are public graphs and Alice knows an isomorphism $\pi : G_0 \rightarrow G_1$.

There is a zero knowledge proof whereby Alice can convince Bob that she knows an isomorphism π from G_0 to G_1, without revealing any information about π.

In particular, she can convince Bob that the graphs really are isomorphic, but Bob cannot turn around and convince Carol of that fact.

Interactive proof of graph isomorphism

	Alice		Bob
1.	Simultaneously choose a random isomorphic copy H of G_0 and an isomorphism $\tau : G_0 \to H$.		
		$\xrightarrow{H}$	
2.		$\xleftarrow{b}$	Choose random $b \in \{0, 1\}$.
3.	If $b = 0$, let $\sigma = \tau$. If $b = 1$, let $\sigma = \tau \circ \pi^{-1}$.	$\xrightarrow{\sigma}$	Check $\sigma(G_b) = H$.

Validity of isomorphism IP

The protocol is similar to the simplified Feige-Fiat-Shamir protocol

If both Alice and Bob follow this protocol, Bob's check always succeeds.

- When $b = 0$, Alice send τ in step 3, and Bob checks that τ is an isomorphism from G_0 to H.
- When $b = 1$, the function σ that Alice computes is an isomorphism from G_1 to H. This is because π^{-1} is an isomorphism from G_1 to G_0 and τ is an isomorphism from G_0 to H. Composing them gives an isomorphism from G_1 to H, so again Bob's check succeeds.

Isomorphism IP is zero knowlege

The protocol is zero knowledge (at least informally) because all Bob learns is a random isomorphic copy H of either G_0 or G_1 and the corresponding isomorphism.

This is information that he could have obtained by himself without Alice's help.

What convinces him that Alice really knows π is that in order to repeatedly pass his checks, the graph H of step 1 must be isomorphic to *both* G_0 and G_1.

Moreover, Alice knows isomorphisms $\sigma_0 : G_0 \to H$ and $\sigma_1 : G_1 \to H$ since she can produce them upon demand.

Hence, she also knows an isomorphism π from G_0 to G_1, since $\sigma_1^{-1} \circ \sigma_0$ is such a function.

FFS authentication and isomorphism IP

We have seen two examples of zero knowledge interactive proofs of knowledge of a secret.

In the simplified Feige-Fiat-Shamir authentication scheme, Alice's secret is a square root of v.

In the graph isomorphism protocol, her secret is the isomorphism π.

In both cases, the protocol has the form that Alice sends Bob a "commitment" string x, Bob sends a query bit b, and Alice replies with a response y_b.

Bob then checks the triple (x, b, y_b) for validity.

Comparison (continued)

In both protocols, neither triple $(x, 0, y_0)$ nor $(x, 1, y_1)$ alone give any information about Alice's secret, but y_0 and y_1 can be combined to reveal her secret.

In the FFS protocol, $y_1 y_0^{-1} \bmod n$ is a square root of v^{-1}. (Note: Since v^{-1} has four square roots, the revealed square root might not be the same as Alice's secret, but it is equally valid as a means of impersonating Alice.)

In the graph isomorphism protocol, $y_1^{-1} \circ y_0$ is an isomorphism mapping G_0 to G_1.

Another viewpoint

One way to view these protocols is that Alice splits her secret into two parts, y_0 and y_1.

By randomization, Alice is able to convince Bob that she really has (or could produce on demand) both parts, but in doing so, she is only forced to reveal one of them.

Each part by itself is statistically independent of the secret and hence gives Bob no information about the secret.

Together, they can be used to recover the secret.

Secret splitting

This is an example of *secret splitting* or *secret sharing*, an important topic in its own right. We have already seen other examples of secret sharing.

In the one-time pad cryptosystem, the message m is split into two parts: the key k are the ciphertext $c = m \oplus k$.

Bob, knowing both k and c, recovers m from by computing $c \oplus k$.

Assuming k is picked randomly, then both k and c are uniformly distributed random bit strings, which is why Eve learns nothing about m from k or c alone.

What's different with zero knowledge proofs is that Bob has a way to check the validity of the parts that he gets during the protocol.

Other Kinds of Interactive Proofs

Other kinds of interactive proofs

Not all interactive proofs follow this simple (x, b, y) pattern.

Suppose Alice wants to prove to Bob that G_0 and G_1 are *non*-isomorphic graphs.

Even ignoring questions of Alice's privacy, there is no obvious data that she can send Bob that will allow him to easily verify that the two graphs are not isomorphic.

However, under a different set of assumptions, Alice can convince Bob that they can't be isomorphic, even though Bob can't do so by himself.

An all-powerful teacher

In this version of interactive proof, we assume that Alice is all-powerful and can compute intractable problems. In particular, given two graphs, she can determine whether or not they are isomorphic.

Bob on the other hand has no extraordinary powers and can just perform computation in the usual way.

Alice uses her computational powers to distinguish isomorphic copies of G_0 from isomorphic copies of G_1. If $G_0 \cong G_1$, there is no way she could do this, since any graph H isomorphic to one of them is also isomorphic to the other.

So by convincing Bob that she is able to reliably distinguish such graphs, she also convinces him that $G_0 \not\cong G_1$.

Interactive proof of graph non-isomorphism

	Alice		Bob
1.			Choose random $b \in \{0,1\}$. Compute a random isomorphic copy H of G_b.
		$\xleftarrow{H}$	
2.	If $H \cong G_0$ let $b' = 0$.		
	If $H \cong G_1$ let $b' = 1$.	$\xrightarrow{b'}$	Check $b' = b$.

Graph non-isomorphism IP is not zero-knowledge

Alice performs a computation for Bob that he could not do himself.

Namely, Alice willingly tells Bob for any H of his choosing whether it is isomorphic to G_0 or to G_1.

(In any implementation of the protocol, she also probably tells him if H is not isomorphic to either one, perhaps by failing in step 2 when b' is undefined.)

Bit commitment

This protocol is an example of *bit-commitment*, another important cryptographic primitive that we will study later.

A bit-commitment is an encryption of a bit with a special property.

- The bit is hidden from anyone not knowing the secret key.
- There is only one valid way of decrypting it, no matter what key is used.

Thus, if $c = E_k(b)$:

- It is hard to find b from c without knowning k.
- For every k', b', if $E_{k'}(b') = c$, then $b = b'$.

In other words, if Bob produces a commitment c to a bit b, then b cannot be recovered from c without knowing Bob's secret encoding key k, but also, there is no key k' that Bob might release that would make it appear that c is a commitment of the bit $1 - b$.

Non-isomorphism protocol viewed as bit commitment

In the non-isomorphism IP, H is a commitment of Bob's bit b.

Suppose Bob gives H to Carol (who doesn't have Alice's extraordinary computational powers).

Later Bob could convince Carol of his bit by telling her the isomorphism that proves $H \cong G_b$.

But there is nothing he could do to make her believe that his bit was really $1 - b$ since $H \ncong G_{1-b}$.

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But there is nothing he could do to make her believe that his bit was really $1 - b$ since $H \not\cong G_{1-b}$.

The actual protocol doesn't use the commitment in quite this way. Rather than having Bob later reveal his bit, Alice uses her special powers to discover the bit committed by H.