Number Theory Summary

Integers Let $\mathbb{Z}$ denote the integers and $\mathbb{Z}^+$ the positive integers.

Division For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, there exist unique integers $q, r$ such that $a = nq + r$ and $0 \leq r < n$. We denote the quotient $q$ by $\lfloor a/n \rfloor$ and the remainder $r$ by $a \mod n$. We say $n$ divides $a$ (written $n \mid a$) if $a \mod n = 0$. If $n \mid a, n$ is called a divisor of $a$. If also $1 < n < |a|$, $n$ is said to be a proper divisor of $a$.

Greatest common divisor The greatest common divisor (gcd) of integers $a, b$ (written $\gcd(a, b)$ or simply $(a, b)$) is the greatest integer $d$ such that $d \mid a$ and $d \mid b$. If $\gcd(a, b) = 1$, then $a$ and $b$ are said to be relatively prime.

Euclidean algorithm Computes $\gcd(a, b)$. Based on two facts: $\gcd(0, b) = b; \gcd(a, b) = \gcd(b, a - bq)$ for any $q \in \mathbb{Z}$. For rapid convergence, take $q = \lfloor a/b \rfloor$, in which case $a - bq = a \mod b$.

Congruence For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, we write $a \equiv b \pmod n$ iff $n \mid (b - a)$. Note $a \equiv b \pmod n$ iff $(a \mod n) = (b \mod n)$.

Modular arithmetic Fix $n \in \mathbb{Z}^+$. Let $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$ and let $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$. For integers $a, b$, define $a \oplus b = (a + b) \mod n$ and $a \otimes b = ab \mod n$. $\oplus$ and $\otimes$ are associative and commutative, and $\otimes$ distributes over $\oplus$. Moreover, $\mod n$ distributes over both $+$ and $\times$, so for example, $a + b \times (c + d) \mod n = (a \mod n) + (b \mod n) \times ((c \mod n) + (d \mod n)) = a \oplus b \otimes (c \oplus d)$. $\mathbb{Z}_n$ is closed under $\oplus$ and $\otimes$, and $\mathbb{Z}_n^*$ is closed under $\otimes$.

Primes and prime factorization A number $p \geq 2$ is prime if it has no proper divisors. Any positive number $n$ can be written uniquely (up to the order of the factors) as a product of primes. Equivalently, there exist unique integers $k, p_1, \ldots, p_k, e_1, \ldots, e_k$ such that $n = \prod_{i=1}^{k} p_i^{e_i}$, $k \geq 0$, $p_1 < p_2 < \ldots < p_k$ are primes, and each $e_i \geq 1$. The product $\prod_{i=1}^{k} p_i^{e_i}$ is called the prime factorization of $n$. A positive integer $n$ is composite if $(\sum_{i=1}^{k} e_i) \geq 2$ in its prime factorization. By these definitions, $n = 1$ has prime factorization with $k = 0$, so 1 is neither prime nor composite.

Linear congruences Let $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}^+$. Let $d = \gcd(a, n)$. If $d \mid b$, then there are $d$ solutions $x$ in $\mathbb{Z}_n$ to the congruence equation $ax \equiv b \pmod n$. If $d \nmid b$, then $ax \equiv b \pmod n$ has no solution.

Extended Euclidean algorithm Finds one solution of $ax \equiv b \pmod n$, or announces that there are none. Call a triple $(g, u, v)$ valid if $g = au + nv$. Algorithm generates valid triples starting with $(n, 0, 1)$ and $(a, 1, 0)$. Goal is to find valid triple $(g, u, v)$ such that $g \mid b$. If found, then $u(b/g)$ solves $ax \equiv b \pmod n$. If none exists, then no solution. Given valid $(g, u, v), (g', u', v')$, can generate new valid triple $(g - qg', u - qu', v - qv')$ for any $q \in \mathbb{Z}$. For rapid convergence, choose $q = \lfloor g/g' \rfloor$, and retain always last two triples. Note: Sequence of generated $g$-values is exactly the same as the sequence of numbers generated by the Euclidean algorithm.
**Inverses** Let \( n \in \mathbb{Z}^+, a \in \mathbb{Z} \). There exists unique \( b \in \mathbb{Z} \) such that \( ab \equiv 1 \pmod{n} \) if \( \gcd(a, n) = 1 \). Such a \( b \), when it exists, is called an inverse of \( a \) modulo \( n \). We write \( a^{-1} \) for the unique inverse of \( a \) modulo \( n \) that is also in \( \mathbb{Z}_n \). Can find \( a^{-1} \mod n \) efficiently by using Extended Euclidean algorithm to solve \( ax \equiv 1 \pmod{n} \).

**Euler function** Let \( \phi(n) = |\mathbb{Z}_n^*| \). One can show that \( \phi(n) = \prod_{i=1}^{k} (p_i - 1)p_i^{e_i-1} \), where \( \prod_{i=1}^{k} p_i^{e_i} \) is the prime factorization of \( n \). In particular, if \( p \) is prime, then \( \phi(p) = p - 1 \), and if \( p, q \) are distinct primes, then \( \phi(pq) = (p - 1)(q - 1) \).

**Euler’s theorem** Let \( n \in \mathbb{Z}^+, a \in \mathbb{Z}_n^* \). Then \( a^{\phi(n)} \equiv 1 \pmod{n} \). As a consequence, if \( r \equiv s \pmod{\phi(n)} \) then \( a^r \equiv a^s \pmod{n} \).

**Order of an element** Let \( n \in \mathbb{Z}^+, a \in \mathbb{Z}_n^* \). We define \( \text{ord}(a) \), the order of \( a \) modulo \( n \), to be the smallest number \( k \geq 1 \) such that \( a^k \equiv 1 \pmod{n} \). Fact: \( \text{ord}(a) | \phi(n) \).

**Primitive roots** Let \( n \in \mathbb{Z}^+, a \in \mathbb{Z}_n^* \). \( a \) is a primitive root of \( n \) iff \( \text{ord}(a) = \phi(n) \). For a primitive root \( a \), it follows that \( \mathbb{Z}_n^* = \{a \mod n, a^2 \mod n, \ldots, a^{\phi(n)} \mod n\} \). If \( n \) has a primitive root, then it has \( \phi(n) \) primitive roots. Primitive roots exist for every prime \( p \) (and for some other numbers as well). \( a \) is a primitive root of \( p \) if \( a^{(p-1)/q} \not\equiv 1 \pmod{p} \) for every prime divisor \( q \) of \( p - 1 \).

**Discrete log** Let \( p \) be a prime, \( a \) a primitive root of \( p \), \( b \in \mathbb{Z}_p^* \) such that \( b \equiv a^k \pmod{p} \) for some \( k, 0 \leq k \leq p - 2 \). We say \( k \) is the discrete logarithm of \( b \) to the base \( a \).

**Chinese remainder theorem** Let \( n_1, \ldots, n_k \) be pairwise relatively prime numbers in \( \mathbb{Z}^+ \), let \( a_1, \ldots, a_k \) be integers, and let \( n = \prod_{i=1}^{k} n_i \). There exists a unique \( x \in \mathbb{Z}_n \) such that \( x \equiv a_i \pmod{n_i} \) for all \( 1 \leq i \leq k \). To compute \( x \), let \( N_i = n/n_i \) and compute \( M_i = N_i^{-1} \mod n_i \), \( 1 \leq i \leq k \). Then \( x = (\sum_{i=1}^{k} a_i M_i N_i) \mod n \).

**Quadratic residues** Let \( a \in \mathbb{Z}^+, n \in \mathbb{Z}^+ \). \( a \) is a quadratic residue modulo \( n \) if there exists \( y \) such that \( a \equiv y^2 \pmod{n} \). \( a \) is sometimes called a square and \( y \) its square root.

**Quadratic residues modulo a prime** If \( p \) is an odd prime, then every quadratic residue in \( \mathbb{Z}_p^* \) has exactly two square roots in \( \mathbb{Z}_p^* \), and exactly half of the elements in \( \mathbb{Z}_p^* \) are quadratic residues. Let \( a \in \mathbb{Z}_p^* \) be a quadratic residue. Then \( a^{(p-1)/2} \equiv (y^2)^{(p-1)/2} \equiv y^{p-1} \equiv 1 \pmod{p} \), where \( y \) a square root of \( a \) modulo \( p \). Let \( g \) be a primitive root modulo \( p \). If \( a \equiv g^k \pmod{p} \), then \( a \) is a quadratic residue modulo \( p \) iff \( k \) is even, in which case its two square roots are \( g^{k/2} \mod{p} \) and \( -g^{k/2} \mod{p} \). If \( p \equiv 3 \pmod{4} \) and \( a \in \mathbb{Z}_p^* \) is a quadratic residue modulo \( p \), then \( a^{(p+1)/4} \) is a square root of \( a \), since \( (a^{(p+1)/4})^2 \equiv aa^{(p-1)/2} \equiv a \pmod{p} \).

**Quadratic residues modulo products of two primes** If \( n = pq \) for \( p, q \) distinct odd primes, then every quadratic residue in \( \mathbb{Z}_n^* \) has exactly four square roots in \( \mathbb{Z}_n^* \), and exactly \( 1/4 \) of the elements in \( \mathbb{Z}_n^* \) are quadratic residues. An element \( a \in \mathbb{Z}_n^* \) is a quadratic residue modulo \( n \) iff it is a quadratic residue modulo \( p \) and modulo \( q \). The four square roots of \( a \) can be found from its two square roots modulo \( p \) and its two square roots modulo \( q \) using the Chinese remainder theorem.

**Legendre symbol** Let \( a \geq 0, p \) an odd prime. \( \left( \frac{a}{p} \right) = 1 \) if \( a \) is a quadratic residue modulo \( p \), \(-1\) if \( a \) is a quadratic non-residue modulo \( p \), and \( 0 \) if \( p | a \). Fact: \( \left( \frac{a}{p} \right) = a^{(p-1)/2} \).
Jacobi symbol  Let $a \geq 0$, $n$ an odd positive number with prime factorization $\prod_{i=1}^{k} p_i^{e_i}$. We define $(\frac{a}{n}) = \prod_{i=1}^{k} \left( \frac{a}{p_i} \right)^{e_i}$. (By convention, this product is 1 when $k = 0$, so $(\frac{a}{1}) = 1$.) The Jacobi and Legendre symbols agree when $n$ is an odd prime. If $(\frac{a}{n}) = -1$ then $a$ is definitely not a quadratic residue modulo $n$, but if $(\frac{a}{n}) = 1$, $a$ might or might not be a quadratic residue.

Computing the Jacobi symbol  $(\frac{a}{n})$ can be computed efficiently by a straightforward recursive algorithm, based on the following identities: $(\frac{0}{n}) = 1$; $(\frac{0}{n}) = 0$ for $n \neq 1$; $(\frac{a_1}{n}) = (\frac{a_2}{n})$ if $a_1 \equiv a_2 \pmod{n}$; $(\frac{2}{n}) = 1$ if $n \equiv \pm 1 \pmod{8}$; $(\frac{2}{n}) = -1$ if $n \equiv \pm 3 \pmod{8}$; $(\frac{2a}{n}) = (\frac{2}{n}) \ (\frac{a}{n})$; $(\frac{a}{n}) = (\frac{n}{a})$ if $a \equiv 1 \pmod{4}$ or $n \equiv 1 \pmod{4}$; $(\frac{a}{n}) = - (\frac{n}{a})$ if $a \equiv n \equiv 3 \pmod{4}$.

Solovay-Strassen test for compositeness  Let $n \in \mathbb{Z}^+$. If $n$ is composite, then for roughly $\frac{1}{2}$ of the numbers $a \in \mathbb{Z}_n^*$, $(\frac{a}{n}) \not\equiv a^{(n-1)/2} \pmod{n}$. If $n$ is prime, then for every $a \in \mathbb{Z}_n^*$, $(\frac{a}{n}) \equiv a^{(n-1)/2} \pmod{n}$.

Miller-Rabin test for compositeness  Let $n \in \mathbb{Z}^+$ and write $n-1 = 2^k m$, where $m$ is odd. Choose $1 \leq a \leq n-1$. Compute $b_i = a^{m2^i} \pmod{n}$ for $i = 0, 1, \ldots, k - 1$. If $n$ is composite, then for roughly $3/4$ of the possible values for $a$, $b_0 \neq 1$ and $b_i \neq -1$ for $0 \leq i \leq k - 1$. If $n$ is prime, then for every $a$, either $b_0 = 1$ or $b_i = -1$ for some $i$, $0 \leq i \leq k - 1$.

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