Throughout this assignment, if a proof or step of a proof follows directly from a definition given or a theorem proven in class or in a reading assignment, then you may simply say that, i.e., you need not reproduce proofs given in class or in the reading.

**Problem 1 (20 points):**
Prove that 2SAT is in NL.

**Problem 2 (15 points):**
Let BIPARTITE be the set of all undirected graphs \( G = (V, E) \) such that \( V \) is the disjoint union \( V = V_1 \sqcup V_2 \) of two vertex sets \( V_1 \) and \( V_2 \), and all edges in \( E \) have one endpoint in \( V_1 \) and one endpoint in \( V_2 \). Prove that BIPARTITE is in NL.

**Problem 3(a) (5 points):**
Let \( k \geq 1 \) be a positive constant. Prove that \( \text{NP} \not\subseteq \text{DTIME}(n^k) \).

**Problem 3(b) (15 points):**
Prove that \( \text{NP}^{\text{EXPCOM}} \not\subseteq \text{DTIME}^{\text{EXPCOM}}(n^k) \). Here, \( \text{EXPCOM} \) is the oracle used in class on Feb. 4, 2016, in the proof of the Baker-Gill-Solovay theorem, and \( \text{DTIME}^A(n^k) \) is the class of sets recognizable by deterministic TMs that run in time \( O(n^k) \) and have access to oracle \( A \).

**Problem 4 (20 points):**
Consider the complexity classes \( \text{DTIME}(n^2) \), \( \text{NTIME}(n^2) \), \( \text{NSPACE}(n^5) \), and \( \text{DSPACE}(n^8) \). For 4 points each, state and prove 5 containment relationships between pairs of these classes.

**Problem 5 (10 points):**
Prove that \( \text{NTIME}(n^k) \not\subseteq \text{PSPACE} \), for any constant \( k \geq 1 \). Does this imply that \( \text{NP} \not\subseteq \text{PSPACE} \)? Briefly justify your answer.

**Problem 6** requires you to work through a proof of the Gap Theorem.

Let \( M_0, M_1, M_2, \ldots \) be an enumeration of all Turing Machines, e.g., the one in Figure 1.7 of your textbook. Let \( \Gamma_i \) be the tape alphabet of \( M_i \).

For integers \( i, k \geq 0 \), define the following property \( P(i, k) \): “Any machine among \( M_0, M_1, \ldots, M_i \), on any input of length \( i \), will halt in fewer than \( k \) steps, halt after more than \( 2^k \) steps, or not halt at all” – that is, on an input of length \( i \), none of these machines will halt immediately after a number of steps that is in the interval \([k, 2^k]\).

Define \( f(i), i \geq 0 \), as follows. Let \( k_0 = 2i \), and, for \( j > 0 \), let \( k_j = 2^{k_{j-1}} + 1 \). We take \( N(i) \) to be \( \sum_{j=0}^{i} |\Gamma_j|^i \), i.e., the total number of inputs of length \( i \) to the first \( i + 1 \) machines \( M_0, \ldots, M_i \). Then \( f(i) \) is defined to be \( k_\ell \), where \( \ell \) is the largest integer less than or equal to \( N(i) \) such that \( P(i, k_\ell) \) is true.
Problem 6(a) (5 points):
Prove that $f$ is well defined, i.e., that there must be an $\ell \leq N(i)$ such that $P(i, k_\ell)$ is true.

Problem 6(b) (10 points):
Prove that any set in $\text{DTIME}(2^{f(n)})$ is in $\text{DTIME}(f(n))$. 