Computer Science 468/568
Homework #6, due in class Thursday, March 29, 2018.

NOTE: in this homework, we consider only Boolean circuits of one output that have fan-in bounded by 2.

1. (10 points) Please list any persons (including course staff) you talked with about this assignment and cite any resources (other than the textbook) you consulted in connection with this assignment (with enough information that another person could find the materials you cite).

2. (30 points) Let $f : \{0,1\}^n \to \{0,1\}$. Give a direct construction to prove that f may be computed by a Boolean circuit of size less than $4 \cdot 2^n$.

   Hint: we can express f as follows.

   $$f(x_1, x_2, \ldots, x_n) = (\neg x_n \wedge f_0(x_1, x_2, \ldots, x_{n-1})) \vee (x_n \wedge f_1(x_1, x_2, \ldots, x_{n-1}))$$

   where $f_0, f_1 : \{0,1\}^{n-1} \to \{0,1\}$ are defined by

   $$f_0(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_{n-1}, 0)$$

   and

   $$f_1(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_{n-1}, 1).$$

3. (30 points)

   (a) Let C be a Boolean circuit. Prove that a path in C from a node of C to the output node can be specified by a string of at most d bits, where d is the depth of C.

   (b) Prove that there exists an algorithm with input $\langle C, x\rangle$ that evaluates Boolean circuit C on input x and uses space $O(\log n + d(C))$, where n is the length of the string specifying $\langle C, x\rangle$ and $d(C)$ is the depth of C. (Hint: be mindful of stack space if you use recursion!)

   (c) Prove that for any $d \geq 1$, if L is a language in UNIFORM $\mathbf{NC}^d$, then $L \in \mathbf{SPACE}(\log^d(n))$.

4. (30 points) Suppose T is a rooted binary tree. Its size is the number of nodes it contains. For any node v of T, let T_v be the subtree of T rooted at v.

   (a) Prove that if T has size S then there is node v of T such that the size of T_v is at least $\frac{S}{3}$ and at most $\frac{2S}{3} + 1$.

(b) Give a Boolean circuit implementing the Boolean functiom $f : \{0,1\}^3 \rightarrow \{0,1\}$ where $f(s,x,y)$ outputs the value of x if $s = 0$ and outputs the value of y if $s = 1$.

(c) Prove that there is a polynomial time algorithm that takes as input an arbitrary Boolean formula ϕ of size n and outputs a Boolean circuit of depth at most $O(\log n)$ that computes the same Boolean function as ϕ. (Hint: the preceding two parts are related to this one.)

5. (30 points) This problem is required of students enrolled in CPSC 568, but not of students enrolled in CPSC 468. Students enrolled in CPSC 468 may do the problem and have it graded, but will receive no extra credit for it.

   Pick out a complexity theory problem (and/or results) relevant to your research (or other) interest and write a proposal of at most 2 pages for a final expository paper of 5-10 pages on this topic. The proposal must include a brief description of your chosen topic, why you are interested in it, and a bibliography of papers and other resources for your topic (which should include at least one original research paper related to your topic.) The proposal and paper must be prepared in a typeset format; latex and pdf are encouraged.

   A complete preliminary draft of your final paper will be due on Thursday, April 19 in class. You will receive feedback and prepare a final draft by 5 pm Thursday, May 3. If you want to discuss your choice of topic, please email Prof. Angluin.