PSPACE-completeness of TQBF

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Quantified Boolean Formula

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n \, \varphi(x_1, x_2, \dots x_n)$$

$$Q_i \in \{\forall, \exists\}$$

e.g.

$$\forall x_1 \exists x_2 \forall x_3 \, ((x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_3))$$

Note: Since there are no unbound variables, a QBF is always either *true* or *false*

TQBF = the set of all true QBFs

TQBF ∈ PSPACE

$$\psi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \, \varphi(x_1, x_2, \ldots x_n)$$

- If $Q_1 = \forall$, recursively check that ψ is true for *both* x_1 = 0 and x_1 = 1
- If $Q_1 = \exists$, recursively check that ψ is true for *either* x_1 = 0 or x_1 = 1
- Reuse space for recursive calls!

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- **Goal**: Define $\psi_{M,x} : \{0,1\}^{2m} \to QBF$ s.t. $\psi_{M,x}(C, C') \in TQBF$ iff there's a path from C to C' in the config. graph of M on input x
- We need to construct a poly-space representation of $\psi_{M,x}(C_{start}, C_{accept})$ in polynomial time

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- $\psi_0(C, C')$ is constructible in poly-space using the techniques of Cook's Theorem
- $\psi_m(C, C')$ = $\psi_{M,x}(C, C')$

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- Inductive definition of ψ_i based on existence of a midpoint of a path (like in Savitch's Theorem):

$$\psi_i(C, C') \Leftrightarrow \exists C''. \psi_{i-1}(C, C'') \wedge \psi_{i-1}(C'', C')$$

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- **Solution**: add universally-quantified variables so that ψ_{i-1} only needs to be mentioned once

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$$\psi_i(C, C') \Leftrightarrow \exists C''. \psi_{i-1}(C, C'') \wedge \psi_{i-1}(C'', C')$$

is equivalent to

$$\psi_i(C, C') \Leftrightarrow \exists C'' \forall D_1 \forall D_2.$$
$$\big((D_1 = C \wedge D_2 = C'') \vee (D_1 = C'' \wedge D_2 = C')\big)$$
$$\Rightarrow \psi_{i-1}(D_1, D_2)$$

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$$\psi_i(C, C') \Leftrightarrow \exists C''. \psi_{i-1}(C, C'') \land \psi_{i-1}(C'', C')$$

is equivalent to

$$\psi_i(C, C') \Leftrightarrow \exists C'' \forall D_1 \forall D_2. \big((D_1 = C \land D_2 = C'') \lor (D_1 = C'' \land D_2 = C')\big) \Rightarrow \psi_{i-1}(D_1, D_2)$$

QED