1 Problem 17.1

Bayes Net

Given a 3CNF formula $x$, let $x^i = x_1 = x_2 = \ldots = x_i = 1$ (i.e. substitute 1 for all $x_1, \ldots, x_i$ and simplify into another 3CNF). Let $f(x)$ be the percentage of satisfying assignments of $x$ where $x_1 = 1$, and $S(x)$ be the total number of satisfying assignments of $x$. Then $f(x) = S(x^1)/S(x)$, and $S(x) = S(x^1)/f(x) = S(x^1)/(f(x)f(x^1)) = \cdots = \Pi_{i=0}^{n}1/f(x^i)$. Both of these formulas are computable in polynomial time given an oracle for $S$ or $f$, respectively, so $FP^{\#SAT} = FP^I$, i.e. $f$ is equivalent to $\#SAT$.

2 Problem 17.2

Computing the Permanent

Given a matrix $M$ with integer entries, interpret it as a complete directed graph $G$ with integer edge weights and self edges. Then $perm(M)$ is the sum of the weights of all cycle covers of $G$. We can now use the construction described on pages 350-351 in the textbook, where integer weights are replaced by parallel edges representing the integer’s decomposition into binary, to obtain a graph $G'$ containing only edge weights in $\{-1, 0, 1\}$ such that, letting $M'$ be the matrix representation of $G'$, we have $perm(M') = perm(M)$. Now we can simply use the formula on page 347 to compute $perm(M')$ by making two calls to our $\#SAT$ oracle. Hence computing the permanent is in $FP^{\#SAT}$.

3 Problem 17.3

XOR Gadget

Consider the XOR gadget on page 349 of the textbook. Label the vertices forming the central diamond $a, b, c, d$ starting from the top and going around in counterclockwise order. Let $C$ be a cycle cover of $G$ with weight $w$. 

**Case 1:** $C$ uses exactly one of $\{(u, u'), (v, v')\}$

**Case 1a:** $C$ uses $(u, u')$ but not $(v, v')$. Since the weight of $(u, u')$ in $G$ is 1, if we take a path from $u$ to $u'$ in $G'$ of length $k$, then the overall cycle cover value is increased by $kw$. We sum this over all cycle covers involving a path from $u$ to $u'$:

1. $\{(u, b, a, c, d, u')\}$; weight = $2w$
2. $\{(u, b, c, a, d, u')\}$; weight = $-w$
3. $\{(u, b, a, d, u'), (c, c)\}$; weight = $w$
4. $\{(u, b, c, d, u'), (a, a)\}$; weight = $2w$
5. $\{(u, b, d, u'), (a, a), (c, c)\}$; weight = $w$
6. $\{(u, b, d, u'), (a, c, a)\}$; weight = $-w$

So the total is indeed $4w$.

**Case 1b:** $C$ uses $(v, v')$ but not $(u, u')$. We look at all the paths from $v$ to $v'$ now:

1. $\{(v, d, c, a, b, v')\}$; weight = $3w$
2. $\{(v, d, a, b, v'), (c, c)\}$; weight = $w$

Once again the total is $4w$ as desired.

**Case 2:** $C$ uses neither of $\{(u, u'), (v, v')\}$. Then any cycle cover in $G'$ must cover $a, b, c, d$ separately. The possibilities are:

1. $\{(a, b, c, d, a)\}$; weight = $-2w$
2. $\{(a, b, d, a), (c, c)\}$; weight = $-w$
3. $\{(a, b, a), (c, d, c)\}$; weight = $6w$
4. $\{(a, b, d, a)\}$; weight = $-3w$

Here the total is 0 as desired.

**Case 3:** $C$ uses both of $\{(u, u'), (v, v')\}$. Looking at the XOR gadget, it's clear that the only way for this to work is if we use all of the edges $(u, b), (b, v'), (v, d), (d, u')$. Thus the possibilities are:

1. $\{(u, b, v'), (v, d, u'), (a, c, a)\}$; weight = $w$
2. $\{(u, b, v'), (v, d, u'), (a, a), (c, c)\}$; weight = $-w$

which again sum to 0.
4 Problem 17.4

#CYCLE Approximation

Suppose we have a TM $M$ that approximates #CYCLE to within a factor of 2 in polynomial time. We will show that we can determine whether a graph $G$ has a Hamiltonian cycle in polynomial time, and so $P = NP$ since this is an NP-complete problem. If $G$ has 4 or fewer vertices, then just brute force check for a Hamiltonian cycle, taking constant time. Otherwise, assume $n > 4$. Let graph $G'$ be as in the proof of Theorem 17.4 in the book, and let $x$ be the number of cycles in $G'$. It’s shown in the book that if $G$ has a Hamiltonian cycle, then $x \geq n^{n^2}$, whereas if $G$ doesn’t have a Hamiltonian cycle, then $x \leq n^{n^2-1}$. Now run $M$ on $G'$ to get output $y$. By our approximation assumption, we know that $y/2 \leq x \leq 2y$. Thus, if $G$ has a Hamiltonian cycle, then $y \geq n^{n^2}/2$, and if $G$ doesn’t have one, then $y \leq 2n^{n^2-1}$. But $n > 4 \Rightarrow 4/n < 1 \Rightarrow 2/n < 1/2 \Rightarrow 2n^{n^2}/n < n^{n^2}/2 \Rightarrow 2n^{n^2-1} < n^{n^2}/2$. Hence the two possible ranges for $y$ are disjoint, and so we can determine whether $G$ has a Hamiltonian cycle in polynomial time.