1. The maximum load is $\Theta(\frac{\ln n}{\ln \ln n})$ *w.h.p.*.

   Upper bound $O(\frac{\ln n}{\ln \ln n})$: Consider such a new process that for each bin (no matter odd or even), choose two random bins uniformly and independently, assign the ball to both bins (we assume we can do this by making a *clone* of the ball). It is obvious that the maximum load of the new process is no smaller than that of the original process introduced in the problem, because the new process can be simulated through running the original process and then adding n additional balls. Observe that the new process is just random $2n$-balls-into-n-bins. By the same analysis as shown in the textbook, the maximum load is $O(\frac{\ln n}{\ln \ln n})$ *w.h.p.*.

   Lower bound $\Omega(\frac{\ln n}{\ln \ln n})$: Consider the loads contributed by even balls (balls that are assigned uniformly and independently at random), which is obviously a lower bound on the final loads. We are thus looking for the lower bound on the maximum load for random $\frac{n}{2}$-balls-into-n-bins. See Appendix A for details.

2. Apply randomized rounding:

$$q_i = \begin{cases} 1 & \text{with prob. } p_i \\ 0 & \text{with prob. } 1 - p_i \end{cases}.$$

   Let $X_{ij} = A_{ij}(p_j - q_j)$, and let $X_i = \sum_j X_{ij}$. Note that

$$E[X_i] = \sum_j A_{ij}(p_i - E(q_i)) = 0.$$

   According to chernoff bound,

$$\Pr[|X_i| > t] \leq 2\exp\left(\frac{-t^2}{2\sum_j 1^2}\right) = 2\exp\left(\frac{-t^2}{2n}\right),$$

   and by union bound,

$$\Pr[\|A(p-q)\|_\infty > t] \leq n\Pr[|X_i| > t] \leq 2n\exp\left(\frac{-t^2}{2n}\right),$$

   which characterizes the concentration of $\|A(p-q)\|_\infty$, in particular, $\|A(p-q)\|_\infty = O(\sqrt{n \ln n})$ *w.h.p.*.

3. Consider the the following LP relaxation of set cover:

$$\begin{aligned} & \min \textstyle\sum_i c_i \\ s.t.: \quad & \textstyle\sum_j M_{ij} c_j \geq 1 \\ & c_i \geq 0. \end{aligned}$$

   Round the c_i to 1 with probability c_i and to 0 otherwise. The probability that S_i is not covered is $\prod_{j \in S_i}(1 - c_j)$, which due to the constraint that $\sum_j M_{ij} c_j \geq 1$ and Lagrange multipliers, is no greater than $(1 - \frac{1}{|S_i|})^{|S_i|} \leq \frac{1}{e}$. Repeat the process untill every set is covered at least once, i.e. c_i has been rounded to 1 for every i. Note that *w.h.p.* it will take at most $O(\ln n)$ iterations, which can be deduced by union bound of failures. After $O(\ln n)$ iterations, by linearity of expectation and union bound,

$$E(|T|) \leq \sum_i O(\ln n)\Pr[c_i \text{ rounded to } 1] = O(\ln n)\sum_i c_i \leq O(\ln n)OPT.$$

   Applying Chernoff bound, we have $|T| = O(\ln n)OPT$ *w.h.p.*.

4. (a) We can construct a set of independent sets of size $c \log n$ with no two intersecting in more than one vertex as follows : we take all independent sets of size $c \log n$. Then we look at all pairs of independent sets of size $c \log n$ and if they intersect in more than one element, we drop one of them from our collection. Thus, we get that $Ef(G)$ is at least : expectation of the total number of independent sets of size $c \log n$, minus the expectation of the number of pairs of independent sets of size $c \log n = r$ which intersect in 2 vertices, minus the expectation of the number of pairs of independent sets of size r which intersect in 3 vertices,

   ....

   minus the expectation of the number of pairs of independent sets of size r which intersect in $r-1$ vertices.

   Each of these expectations can be written as the sum of expectation of indicator random variables (one for each set in the first expression and one for each pair of sets in the others) in the standard way. Thus,

$$\begin{aligned} E(f(G)) &\geq \binom{n}{r}(1-p)^{\binom{r}{2}} - \sum_{l=2}^{r-1} \binom{n}{r}\binom{r}{l}\binom{n-r}{r-l}(1-p)^{2\binom{r}{2}-\binom{l}{2}} \\ &\geq \binom{n}{r}(1-p)^{\binom{r}{2}} \left[1 - \sum_{l=2}^{r-1} \binom{r}{l}\binom{n}{r-l}(1-p)^{\binom{r}{2}-\binom{l}{2}}\right]. \end{aligned}$$

   We choose r so that $\binom{n}{r}(1-p)^{\binom{r}{2}} = n^{5/3}$. Note that $r = O(\log n)$ since p is constant. Now, consider the first term in the sum on the r.h.s. of the above :

$$\binom{r}{l}\binom{n}{r-2}(1-p)^{\binom{r}{2}-1}.$$

   The leading term here is $\binom{n}{r-2}$ which is easily seen to be $O(r^2/n^2)$ times $\binom{n}{r}$; from this argue that for $l = 2$, $\binom{r}{l}\binom{n}{r-l}(1-p)^{\binom{r}{2}-\binom{l}{2}} \in O(1/\sqrt{n})$. The other terms are even smaller, so the sum is $o(1)$ proving what we want.

   (b) Consider the edge exposure martingale formed by $f(G)$. Note that exposing a single edge can change the value of $f(G)$ by at most 1. Apply the Azuma's lemma:

$$\Pr[f(G)=0] \leq \Pr[|f(G) - E(f(G))| \geq E(f(G))] \leq 2\exp\left(\frac{-E^2(f(G))}{2\binom{n}{2}}\right) \leq \exp(-\Omega(n^{4/3})).$$

   (c) Routine.

   (d) The idea is to repeatedly choose a $\Omega(\log n)$ independent set and assign with a new color, until reaching the trivial case, that is, $O(n/\log n)$ vertices remain uncolored, and then assign each uncolored vertex with a new color. Note that the probability of the failure of each iteration is exponentially small, thus the overall failure is small due to union bound, i.e. *w.h.p.*, there is a $O(n/\log n)$-coloring of the random graph.

5. (a) Assume k is drawn from the distribution that

$$\Pr[k=i] = \frac{(t\delta)^i}{i! e^{t\delta}},$$

   where t is chosen to minimize $\frac{E[e^{t|X|}]}{e^{t\delta}}$. Note that $\sum_i \Pr[k=i] = 1$, i.e. the distribution is well-defined.

$$E\left[\frac{E\left[|X|^k\right]}{\delta^k}\right] = \sum_i \frac{E\left[|X|^i\right]}{\delta^i}\Pr[k=i] = \frac{E\left[e^{t|X|}\right]}{e^{t\delta}} = \min_{t\geq 0}\frac{E\left[e^{t|X|}\right]}{e^{t\delta}},$$

   i.e. there exists a k that $\frac{E\left[|X|^k\right]}{\delta^k} \leq \min_t \frac{E\left[e^{t|X|}\right]}{e^{t\delta}}$.

   (b) We do not use kth moment bound because we do not know for which value of k the kth moment bound is better, and to find it is a non-trivial task by itself.

A Lower bound of balls-into-bins

We assume that $m = \Theta(n)$. Let X_i be the random variable that indicates whether the load of i^{th} bin is greater than some threshold t, where $t = \Theta(\log n / \log\log n)$. Let $X = \sum_i X_i$. Recall that the load of a specific bin follows the binomial distribution $B(n, \frac{1}{m})$. It is easy to verify that

$$E(X) = n \cdot \Pr\left[B\left(n, \frac{1}{m}\right) > t\right] = \omega(1),$$

for appropriate $t = \Theta(\log n / \log\log n)$.

It is obvious that X_i and X_j are negatively correlated for any $i \neq j$, thus $cov(X_i, X_j) \leq 0$. Therefore due to chebychev's inequality,

$$\Pr[\text{max load is smaller than } t] \leq \Pr[X = 0] \leq \frac{var(X)}{E^2(X)} = \frac{\sum_i var(X_i) + \sum_{i \neq j} cov(X_i, X_j)}{E^2(X)} \leq \frac{1}{E(X)} = o(1).$$