1. Let S be the subset of V that $\Phi(S) = \Phi$ and $|S| \le \frac{1}{2}|V|$. Because the M.C. is lazy, symmetric and irreducible, its stationary distribution is uniform. Therefore,

$$\Phi = \Phi(S) = \frac{\sum_{i\in S, j\notin S} P_{ij}/|V|}{|S|/|V|} = \frac{\sum_{i\in S,j\notin S} P_{ij}}{|S|}.$$

According to Courant-Fisher inequality,

$$1 - \lambda_2 = \min_{x\perp 1} \frac{\sum_{i<j} P_{ij}(x_i - x_j)^2}{\sum_i x_i^2}.$$

Let y be a vector such that $y_i = |\bar{S}|$ if $i \in S$ and $y_i = -|S|$ if otherwise, where $\bar{S}$ denotes the complement of S. It is easy to verify that y is orthogonal to 1. Therefore,

$$\begin{aligned} 1 - \lambda_2 &\le \frac{\sum_{i<j} P_{ij}(y_i - y_j)^2}{\sum_i y_i^2} \\ &= \frac{\sum_{i\in S,j\notin S} P_{ij}(|S| + |\bar{S}|)^2}{|S|^2|\bar{S}| + |S||\bar{S}|^2} \\ &= \frac{\sum_{i\in S,j\notin S} P_{ij}|V|}{|S||\bar{S}|} \\ &\le \frac{2\sum_{i\in S,j\notin S} P_{ij}}{|S|} \\ &= 2\Phi. \end{aligned}$$

2. Since the M.C. is lazy, symmetric, and irreducible, by the same argument in (1), we have that,

$$\Phi(W) = \frac{\sum_{i\in W,j\notin W} P_{ij}}{|S|} = \frac{|\partial W|}{2d \cdot |W|},$$

where ∂W denotes the *edge boundary* of W, i.e. $\partial W = \{\{u, v\} \mid u \in W, v \notin W\}$. Since the graph G is d-regular, it holds that $N(W) \ge |\partial W|/d$. Therefore,

$$N(W) \ge |\partial W|/d = 2\Phi(W)|W| \ge 2\Phi|W| \ge (1 - \lambda_2)|W|.$$

3. Let $K^+ = \{x \in K \mid f(x) \ge 0\}$ and $K^- = \{x \in K \mid f(x) < 0\}$. They are both convex since they are intersections of convex bodies. Let

$$K_*^+ = \{(x, y) \mid x \in K^+, 0 \le y \le f(x)\}$$

and

$$K_*^- = \{(x, y) \mid x \in K^-, f(x) \le y \le 0\}.$$

Let (x, y) and (x', y') be two vectors in K_*^+. Since K^+ is convex, $\frac{x+x'}{2} \in K^+$. Due to the linearity of f, $\frac{y+y'}{2} \le \frac{f(x)+f(x')}{2} = f(\frac{x+x'}{2})$. Therefore, $(\frac{x+x'}{2}, \frac{y+y'}{2}) \in K_*^+$, i.e. K_*^+ is convex. Similarly, K_*^- is also convex. Therefore $vol(K_*^+)$, $vol(K_*^-)$ and $vol(K)$ are all estimable. So is $\frac{\int_K f(x)dx}{\int_K dx}$, which is equal to $\frac{vol(K_*^+) - vol(K_*^-)}{vol(K)}$.

4. To prevent trivializing the problem, we assume that the transition probability is uniform among all neighbors for any vertex.

   The maximum conductance is achieved via complete graph, where the conductance is $\Theta(1)$. The minimum conductance is achieved via dumbbell graph (two equal size complete components connected by a single edge), where the conductance is $O(\frac{1}{n^2})$.

5. (a) I will just give a sketch of the proof. Students may work out the details.

We adopt the *canonical path* technique from the book, namely, for every pair of spanning trees T_s and T_t, we construct a path from T_s to T_t as well as a dual path from T_t to T_s. The paths are so constructed that for any particular transition $S \to T$, the number of paths that pass through it is bounded.

The key step is to figure out a construction that satisfies the *duality condition*, namely, for the i^{th} tree T_i in the path from T_s to T_t, and the the i^{th} tree T_i' (the dual tree of T_i) in the dual path from T_t to T_s, it holds that $T_i \cup T_i' = T_s \cup T_t$ and $T_i \cap T_i' = T_s \cap T_t$. Because of this symmetric structure of the canonical paths, for any particular transition $S \to T$, any path through the transition can be uniquely specified by (T', i), where T' is the dual tree of T in the dual path, and T is the i^{th} tree in the path (students may verify this in detail). Therefore, for any transition, the number of paths through it is at most nN where N is the number of spanning trees. According to the same argument as in the text book, the conductance is at least $n^{-O(1)}$.

(b) A $n^{-O(1)}$ conductance implies a polynomial time algorithm that samples a spanning tree uniformly at random with exponentially small bias.

Let e be a specific edge in the graph G. Let N_e be the number of spanning trees that contain e and let $N_{\bar{e}}$ be the number of spanning trees that do not contain e. Since $N_e + N_{\bar{e}} = N$ is the total number of spanning trees, one of the ratios N/N_e and $N/N_{\bar{e}}$ is at most 2, which can be estimated to desired precision within polynomial time via random sampling. Note that N_e is the number of spanning trees for the graph resulting from contracting e, and $N_{\bar{e}}$ is the number of spanning trees for the the graph $G \setminus \{e\}$. Both of them decrease the size of the problem instance, thus can be estimated recursively.