Game Positions

Game position = set of positions one can move to

In traditional 1-row Nim

0 stars

$\sqcup = \{\ \} = \emptyset$

$0 = \{\sqcup\}$

$00 = \{\sqcup, 0\}$

$000 = \{\sqcup, 0, 00\}$

$0000 =$

$\vdots$

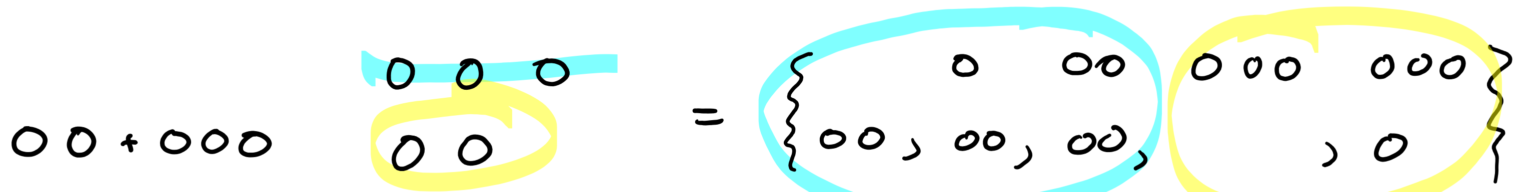
$0\ 0\ 0\ 0\ \cancel{0}$

take 1, 2, or 3

N : next player has winning strategy/move

P : prev player has winning strategy

Sums of Games



$$G + H = \{ G' + H \text{ s.t. } G' \text{ is an option of } G \} \cup \{ G + H' \text{ s.t. } H' \text{ is an option of } H \}$$

position reachable in one move

OOO
OO
OOOO

standard + combine 2 rows
not a sum

x   xxxx   xxx

x + xxxx + xxx

Equivalence of Games

For impartial, normal games G, G', say $G \approx G'$ if and only if
equivalent

for every game H $G+H$ and $G'+H$ have same outcome class (both N or both P)

Is $*2 \approx *1$? NO

Nimbers

$*2 + *2$ P

$*1 + *2$ N

can move to $*1 + *1$

Is $*5 \approx *3$? NO

$*5 + *3$

can move to $*3 + *3$
P

so N

$*3 + *3$

P

Conjecture: $\forall m, n \in \mathbb{N},\ m \neq n \rightarrow *m \not\approx *n$
(True)

Is $*2 + *1 \approx *3$
YES

$*2 + *1 + *0$ N $*3 + *0$ N

$*2 + *1 + *1$ N $*3 + *1$ N

$*2 + *1 + *2$ N $*3 + *2$ N

$*2 + *1 + *3$ P $*3 + *3$ P

$*2 + *1 + *4$ $*3 + *4$

$$\vdots$$

Conjecture: $*n + *m \approx *(n \oplus m)$

bitwise exclusive or

$$\begin{array}{cc} 2 & 10 \\ 1 & \underline{01} \\ 3 & 11 \end{array}$$

Properties of Equivalence

$G = G + \underline{*0}$ has same outcome as $H + \underline{*0} = H$

For all finite, impartial, normal games G, H, K

$G \approx H \longrightarrow$ G, H have same outcome class

$G + H = G + H$

$G \approx G$ reflexive

$G \approx H \longrightarrow H \approx G$ symmetric

$G \approx H$ and $H \approx K \longrightarrow G \approx K$ transitive

$G + H \approx H + G$ commute

$\underline{(G+H)} + K \approx G + (H + K)$ associative

Lemmas

L1: Any position $G+H$ is an N position if G, H are in different outcome classes and is a P position if G, H are both P positions.

$N+P = N$
$P+N = N$
$P+P = P$
$N+N = ?$

Proof: (Ind. on length of $G+H$)
max moves until terminal

Base case: $(n=0)$ $G = H = \{\}$ P pos $G+H = \{\}$ P pos

Ind step: Suppose L1 holds for sums of $<k$ for some $k>0$

Let $G+H$ be a sum of length k

case 1: G is N, H is P

G has option G' s.t. G' is P pos

$G'+H$ has length $<k$ so ind. hyp. applies

$P+P=P$

so $G'+H$ is P; is an option of $G+H$, so $G+H$ is N position

case 2: G is P, H is N similar

case 3: G is P, H is P

consider all options of $G+H$: $\underset{\substack{N+P\\N}}{G'+H}$ or $\underset{\substack{P+N\\N}}{G+H'}$

option of G — option of H'

apply ind. hyp:

all options of $G+H$ are N, so $G+H$ is P

L2: For every P position A and every position G, $G+A \approx G$

Proof: Suppose A is a P position and G is any position

Let H be any game [want: $G+A+H$, $G+H$ have same outcome class]

2 cases: 1) $G+H$ is P then $G+A+H = \underbrace{G+H}_{P} + \underbrace{A}_{P}$ is P

2) $G+H$ is N then $G+A+H = \underbrace{G+H}_{N} + \underbrace{A}_{P}$ is N

L3: $G \approx G'$ if and only if $G+G'$ is a P position

Proof: $\rightarrow$: Suppose $G \approx G'$. Then $G+G$, $G'+G$ have same outcome class

$G+G$ is P, so $G'+G$ is too

$\leftarrow$: Suppose $G+G'$ is a P position

Then $G+(G+G') \approx G$ (L2 with $A = G+G'$)

and $G'+(G+G) \approx G'$ (L2 with $A = G+G$)

$\approx$ is associative and commutative

So $G \approx G+(G+G') \approx G'+(G+G) \approx G'$

and $G \approx G'$ ($\approx$ is transitive)

L4: If $G = \{G_1, \dots, G_n\}$ and $G_i \approx G_i'$

then $G \approx \{G_1', \dots, G_n'\}$

Proof: [use L3: show $G + \{G_1', \dots, G_n'\}$ is a P position]

Consider all options of $G + \underbrace{\{G_1', \dots, G_n'\}}_{\text{call this } S}$

They are of 2 forms: 1) $G_i + \{G_1', \dots, G_n'\}$, which itself has option $G_i + G_i'$ which is P b/c $G_i \approx G_i'$

So $G_i + \{G_i' + G_n'\}$ is N

2) $G + G_i'$, which has option $G_i + G_i'$, which is P

So $G + G_i'$ is N

All options of $G + \{G_1', \dots, G_n'\}$ are N so it is P

so $G \approx \{G_1', \dots, G_n'\}$ by L3

THM (Sprague-Grundy) Every finite, impartial normal game is equivalent to some nimber.

Proof:

$G \approx *0$

COR: Let $G = \{\overbrace{G_1, \dots, G_m}^{*3,\ *4,\ *1}\}$ where $G_1 \approx *n_1, \dots G_m \approx *n_m$

then $G \approx \operatorname{mex}(n_1, \dots, n_m)$

minimum excludant (smallest nonneg integer not in set)

COR: If $G_1 \approx *n_1$ and $G_2 \approx *n_2$ then $G_1 + G_2 \approx *(n_1 \oplus n_2)$

Proof:

Kayles (long way)

$\cup = \{\ \} = *0$ $*0$

$x = \{\cup\} \approx \{*0\}$ $\mathrm{mex}(\{0\})=1$ $*1$

incr len

$xx = \{x, \cup\} \approx \{*1, *0\}$ $\mathrm{mex}(\{0,1\})=2$ $*2$

$x\ \ x = \{x\} \approx \{*1\}$ $\mathrm{mex}(\{1\})=0$ $*0$

$xxx = \{x, xx, x\ \ x\} \approx \{*1, *2, *0\}$ $*3$

$xx\ \ x = \{x, x\ \ x, xx\} \approx \{*1, *0, *2\}$ $*3$

$x\ \ x\ \ x = \{x\ \ x\} \approx \{*0\}$ $*1$

xxxx
x xxx
x xx x
x x x x
xx xx
xx x x

$\cup$ $*0$

$x = \{\cup\} \approx \{*0\}$ $*1$

$xx =$ $*2$

xxx $*3$

$xxxx = \{xxx, xx, xx\ \ x, x\ \ x\}$ $*1$

$*3$, $*2$, $xx + x$: $*2 + *1$, $*(2 \oplus 1)$, $*3$; $x + x$: $*1 + *1$, $*(1 \oplus 1)$, $*0$

0 1 2 3 4 5 6 7 8 9 10 11 12

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

xxxxxxx   xxx~~xx~~   xxxx

$x7 + x5 + x4$

$\approx *2 + *4 + *1$ $\to *3$

$*3$

010
100
001
———
111

111
100
———
011

$*2 + *3 + *1$