Kayles (long way)

$\emptyset = \{\ \} = *0$ — $*0$

$x = \{\emptyset\} \simeq \{*0\}$ mex$(\{0\})=1$ — $*1$

$xx = \{x, \emptyset\} \simeq \{*1, *0\}$ mex$(\{0,1\})=2$ — $*2$

$x\ \ x = \{x\} \simeq \{*1\}$ mex$(\{1\})=0$ — $*0$

$xxx = \{x, xx, x\ \ x\} \simeq \{*1, *2, *0\}$ — $*3$

$xx\ \ x = \{x, x\ \ x, xx\} \simeq \{*1, *0, *2\}$ — $*3$

$x\ \ x\ \ x = \{x\ \ x\} \simeq \{*0\}$ — $*1$

incr len

xxxx
x xxx
x xx x
x x x x
xx xx
xx x x

$\emptyset$ — $*0$

$x = \{\emptyset\} \simeq \{*0\}$ — $*1$

$xx =$ — $*2$

xxx — $*3$

$xxxx = \{xxx, xx, xx\ \ x, x\ \ x\}$ — $*1$

$*3$, $*2$, $xx + x$: $*2+*1$, $*(2+1)$, $*3$; $x + x$: $*1+*1$, $*(1\oplus1)$, $*0$

$xxxxx = \{xxxx, x\ \ xxx, x\ \ xx, xx\ \ x, x\ \ xx\}$ $*4$

$*1$, $*1 + *3$, $*(1\oplus3) = *2$, $*0$, $*3$, $*1$, $*2$, $*(1\oplus2) = *3$

0 1 2 3 4 5 6 7 8 9 10 11 12

0, 1, 2, 3, 1, 4, 3, 2, 1, 4, 2, 6, 4, 1, 2, 7, 1, 4, 3, 2, 1, 4, 6, 7, 4, 1, 2, 8, 5, 4, 7, 2, 1, 8, 6, 7

xxxxxxx   ~~xxxxx~~   xxxx

$x7 + x5 + x4$

$\simeq *2 + *4 + *1$ → $*3$

10x

$9x + 1x \simeq *4 + *1 \simeq *(4+1) = *5$

$8x + 2x$

xxxxxxx   xxx   xxxx xxxxxxx

$*2$ $*3$ $*3 + *2 \simeq *(3\oplus2) = *1$ $*6$

010
100
001
111

111
100
011

$*2 + *3 + *1$

00
000
0 ~~0 0 0 0 0~~

010
011
110
111

010
011
001
000

111
110

$$\begin{array}{r} 111 \\ 110 \\ \hline 001 \end{array}$$

start of game

we chose 66 ← 1 3 3 6 6 ?

end of 1st turn

we chose 6 ← 1 6 6 6 6

used 6 (24 up) and 4 (50)

kept 555 ← 4 4 5 5 5

same as ↗ +

← 1 4 5 5 5

Every finite, impartial normal game is equivalent to some nimber.

Proof: (ind. on length of game)

Base case ($n=0$): only game of len 0 is $\{\} = *0 \approx *0$

Induction step: Let G be a game of length $k>0$ and suppose all games G' of length $<k$ are equivalent to some nimber.

Write $G = \{G_1, \ldots, G_\ell\}$ where $len(G_i) < len(G) = k$ for all i

So by induction hypothesis,

$G_1 \approx *n_1,\ G_2 \approx *n_2, \ldots, G_\ell \approx *n_\ell$ for some $n_1, n_2, \ldots, n_\ell$

$\underline{0, 1}, \underline{3, 4}$ $m=2$

so $G \approx \underbrace{\{*n_1, *n_2, \ldots, *n_\ell\}}_{G'}$ (L4)

Claim: $G' + *m$ is P-pos where $m = mex(\{n_1, \ldots, n_\ell\})$

so $G' \approx *m$ (use L3)

Consider all options of $G' + *m$

Three cases: i) $G' + *j$, where $j < m$

G' has option $*j$ (one of the $n_i = j$)

so $G' + *j$ has option $*j + *j$, P pos (mex)

so $G' + *j$ is N pos

want all 3 cases to be N (winning pos)

ii) $*i + *m$, where $i < m$

and so $*i + *m$ has option $*i + *i$ which is P

so $*i + *m$ is N

iii) $*i + *m$, where $i > m$

so $*i + *m$ has option $*m + *m$, which is P

$*i + *m$ is N

iv) ~~$*i + *m$~~, $i=m$

All options of $G' + *m$ are N-positions

So $G' + *m$ in a P position (def)

$\therefore G' \approx *m$ (L3)

$G \approx G'$

$G \approx *m$ (transitivity)

Nim-sum

Theorem : $*n + *m \approx *(n \oplus m)$

Proof: (induction on length of game, $n+m$)

Base case ($n+m=0$): Then $n=0$, $m=0$, $n \oplus m = 0$

$*n + *m = *0 + *0 = \{\} = *0$ $\qquad 0 \oplus 0 = 0$

Induction Step: Suppose $n+m>0$ and all n', m' s.t. $n'+m' < n+m$ have $*n' + *m' \approx *(n' \oplus m')$

$$*n + *m = \left\{ \begin{array}{l} *0 + *m,\ *1 + *m,\ \ldots,\ *(n-1) + *m, \\ *n + *0,\ *n + *1,\ \ldots,\ *n + *(m-1) \end{array} \right\}$$

$$\approx *\left(\operatorname{mex}\left\{ \begin{array}{l} 0 \oplus m,\ 1 \oplus m,\ \ldots,\ (n-1) \oplus m, \\ n \oplus 0,\ n \oplus 1,\ \ldots,\ n \oplus (m-1) \end{array} \right\}\right)$$

$x = (m \oplus x) \oplus m$

Claim: $\operatorname{mex}(\{0 \oplus m, \ldots, (n-1) \oplus m, n \oplus 0, \ldots, n \oplus (m-1)\}) = n \oplus m$

1) $n \oplus m$ is excluded from

suppose $i \oplus m = n \oplus m$ for $i < n$

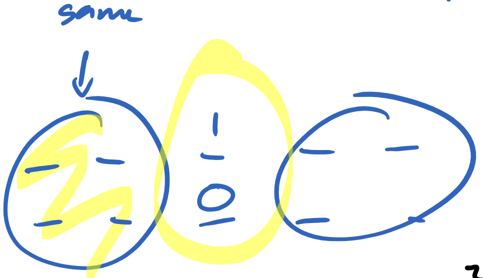
then $i \oplus m \oplus m = n \oplus m \oplus m$

$i \oplus 0 = n \oplus 0$

$i = n$ $\Rightarrow\Leftarrow$

similar

same



2) All x s.t. $0 \le x < n \oplus m$ are included:

Let x be $0 \le x < n \oplus m$

Find most significant bit where x, $n \oplus m$ differ

That bit is 1 in $n \oplus m$ and 0 in x

To be 1 in $n \oplus m$, corresponding bits in n, m are 0, 1

Assume, WLOG, bits are 1 in n, 0 in m

n	d_1	d_2	…	1	…
m	e_1	e_2	…	0	…
$n \oplus m$	a_1	a_2	…	1	… b_k
x	a_1	a_2	…	0	… c_k
$m \oplus x$	f_1	f_2		0	

$f_i = a_i \oplus e_i$ $\qquad a_i = d_i \oplus e_i$

$= d_i \oplus e_i \oplus e_i$

$= d_i$

So $m \oplus x < n$

and $*(m \oplus x) + *m$ is an option of $*n + *m$

But $*(m \oplus x) + *m \approx *(m \oplus x \oplus m)$

$= *x$

Markov Decision Process

Markov Decision Process : (S, A, p, R)

states (positions)

rewards

probability

actions (moves)

$$p : \underbrace{S}_{\text{state}} \times \underbrace{A}_{\text{action}} \times \underbrace{S}_{\text{new state}} \times \underbrace{R}_{\text{reward}} \to [0, 1]$$

$$p\left(\begin{matrix}\text{initial state} \\ \text{w/ 1st rolled} \\ 16666\end{matrix}, \text{keep 4 6's}, \begin{matrix}\text{initial} \\ \text{state after} \\ \text{1st reroll} \\ 56666\end{matrix}, 0\right) = \frac{1}{6}$$

$$p\left(\begin{matrix}\text{initial state} \\ \text{w/ 1st roll} \\ 16666\end{matrix}, \text{keep 4 6's}, \begin{matrix}\text{end of} \\ \text{game}\end{matrix}, 0\right) = 0$$

because you can't do this in one step

because you never get 39 points when rolling

$$p\left(\begin{matrix}\text{initial state} \\ \text{w/ 1st rolled} \\ 16666\end{matrix}, \text{keep 4 6's}, \begin{matrix}\text{initial} \\ \text{state after} \\ \text{1st reroll} \\ 56666\end{matrix}, 39\right) = 0$$