Simultaneous Play Games (matrix games)

payoff for I, payoff for II

		II	
I	R	P	S
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

A = matrix of I's payoffs

$$A=\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

B = matrix for II

$$B=\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$v^- < v^+$

H P(win)=.67

$a_{ij} =$

Penalty Kick

kicker \ goal keeper	L	R
L	$\frac{1}{2}, -\frac{1}{8}$	$\frac{1}{4}, -\frac{1}{4}$
R	$\frac{1}{4}, -\frac{1}{4}$	$\frac{1}{6}, -\frac{1}{6}$

1-0 38:00

2-0 P(win)=.91

1-0 P(win)=.68

zero-sum: $a_{ij}+b_{ij}=0$ for all i,j

constant-sum $a_{ij}+b_{ij}=C$ for all i,j

zero-sum → 1) play game $A-C$

2) award I additional payoff C

$A+B=C$

$(A-C)+B=0$

no strategy

neither player has incentive to unilaterally change

I \ II		W	X	Y	Z
A	$\frac{1}{2}$	$\frac{3}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
B	-1	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$
C	-1	$\frac{1}{2}$	0	-1	$1\frac{1}{2}$
		$\frac{3}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$

$v^- = v^+$

v^- : payoff that P1 can guarantee

: max of row mins

v^+ : payoff that P2 can guarantee

: min of col maximums

Stag Hunt

I	S	H
Stag	2,2	0,1
Hare	1,0	1,1

$R > T = P > S$

Prisoners Dilemma

I	Coop	Defect
Coop	3,3	0,5
Defect	5,0	1,1

not equilibrium

equilibrium

R = reward for cooperating

P = punishment for defecting

S = sucker's reward

T = reward for temptations

$T > R > P > S$

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

	L	R
L	$\frac{1}{8}$	$\frac{1}{4}$
R	$\frac{1}{4}$	$\frac{1}{6}$

	W	X	Y	Z
A	$\frac{3}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
B	-1	$\frac{1}{3}$	$\frac{1}{4}$	$-\frac{1}{2}$
C	$\frac{1}{2}$	0	-1	$\frac{1}{2}$

For any constant-sum game, $v^- \le v^+$

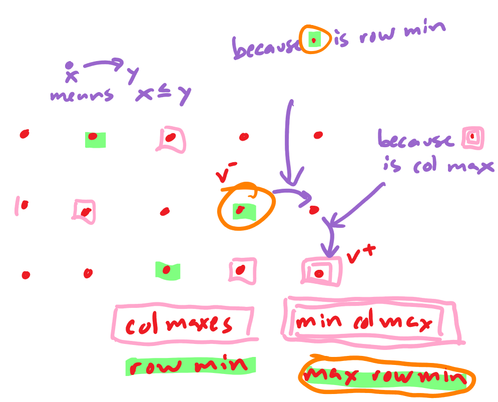
$\forall_{i,j}$ $v^- = \min_{j'} a_{ij'} \le a_{ij}$ (def min)

$\forall_j$ $\max_i \min_{j'} a_{ij'} \le \max_i a_{ij}$

$\cancel{\min_j}$ $\max_i \min_{j'} a_{ij'} \le \min_j \max_i a_{ij}$

$\max_i \min_j a_{ij} \le \min_j \max_i a_{ij}$

$v^- \le v^+$



Equilibrium: where neither player has incentive to unilaterally change strategy (saddle point)

A constant-sum game A has an equilibrium in pure strategies if and only if $v^- = v^+$

$\Rightarrow$: Suppose A has an equilibrium in pure strategies.

Then $\exists\ i^*, j^*$ s.t. $a_{ij^*} \le a_{i^*j^*}$ for all i $\quad a_{i^*j^*} \le a_{i^*j}$ for all j (def. equilibrium)

so $\max_i a_{ij^*} \le a_{i^*j^*}$ for all i $\quad a_{i^*j^*} \le \min_j a_{i^*j}$ for all j

bigger than all terms in max $\quad$ smaller than all terms in min

and $\min_j \max_i a_{ij} \le \max_i a_{ij^*}$ $\quad \min_j a_{i^*j} \le \max_i \min_j a_{ij}$

term is in min (when $j=j^*$) $\quad$ term is in max (when $i=i^*$)

so $v^+ \le \max_i a_{ij^*} \le a_{i^*j^*} \le \min_j a_{i^*j} \le v^-$

but $v^- \le v^+$ so all $\le$ are $=$ (squeeze)

$\Leftarrow$ Suppose $v^- = v^+$. Let i^* be the i s.t. $v^- = \max_i \min_j a_{ij}$

j^* be the j s.t. $v^+ = \min_j \max_i a_{ij}$

for any i,j $\quad \underbrace{a_{i^*j}}_{\text{in the min}} \ge \min_j a_{i^*j} = v^+ = v^- = \max_i a_{ij^*} \ge \underbrace{a_{ij^*}}_{\text{in the max}}$

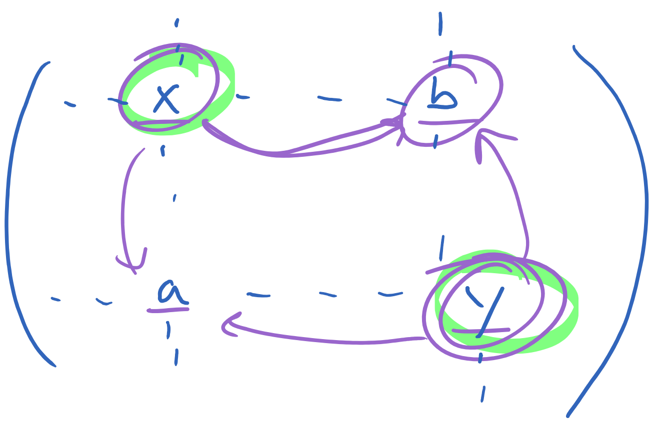
$a_{i^*j^*} \ge v^+ = v^- \ge a_{i^*j^*}$ $\quad$ the above works for any i,j, including $i=i^*$ and $j=j^*$

and $\ge$ are $=$

$\therefore\ v^+ = v^- = a_{i^*j^*}$

$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}$ for all i,j

Suppose there are 2 equilibria in pure strategies (i_1, j_1) and (i_2, j_2)
with values $a_{i_1j_1} = v_1$ $\quad a_{i_2j_2} = v_2$



$x \le b \le y \le a \le x$

squeeze! all $\le$ are $=$

$\underset{v_1}{x} = b = \underset{v_2}{y} = a = x$

2 equilibria have same payoff

Mixed Strategies - probability distribution over actions

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

X = mixed strat for I $(x_1\ x_2\ \dots\ x_n)$

$x_1 + \dots + x_n = 1$

$Y = (y_1\ y_2\ \dots\ y_m)$ $\quad 0 \le x_i \le 1$

equilibrium

$$X^* = \left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right) = Y^*$$

$$E(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i,\ \text{II plays } j)\cdot a_{ij}$$

expected payoff for I

$$\sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i)\cdot P(\text{II plays } j)\cdot a_{ij}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\cdot x_i\cdot y_j$$

$$= X A Y^T \quad 3\times 3$$

$$\left(\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}\right)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$\tfrac{1}{2}\cdot 0 + \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot -1$

$\tfrac{1}{2}\cdot -1 + \tfrac{1}{4}\cdot 0 + \tfrac{1}{4}\cdot 1 = -\tfrac{1}{4}$

$\tfrac{1}{2}\cdot 1 + \tfrac{1}{4}\cdot -1 \cdot \tfrac{1}{4}\cdot 0$

$$X = \left(\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{4}\right)$$

$$Y = \left(\tfrac{1}{3}\ 0\ \tfrac{2}{3}\right)$$

$$\left(\tfrac{1}{2}\ \tfrac{1}{4}\ \tfrac{1}{4}\right)\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}$$

$$\begin{array}{ccc} \frac{1}{3} & 0 & \frac{2}{3} \end{array}$$

$$= \left(0\ \ -\tfrac{1}{4}\ \ \tfrac{1}{4}\right)\cdot\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{1}{6}$$

expected payoff for I if they play X and II plays col 1

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

A pair of mixed strategies X^*, Y^* is an equilibrium if

$$E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y) \quad \text{for all } X, Y$$

Every game has an equilibrium in mixed strategies

Equilibrium Theorem: If X^*, Y^* is an equilibrium in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

$$E(X^*, j) = E(i, Y^*) = \text{value}(A)$$