Mixed Strategies

probability distribution over actions

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$X =$ mixed strat for I $(x_1, x_2 \dots x_n)$

$x_1 + \dots + x_n = 1$

$Y = (y_1\ y_2 \dots y_m)$ $\quad 0 \le x_i \le 1$

equilibrium

$X^* = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}) = Y^*$

$$E(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i, \text{ II plays } j)\cdot a_{ij}$$

expected payoff for I

$$\sum_{i=1}^{n}\sum_{j=1}^{m} P(\text{I plays } i)\cdot P(\text{II plays } j)\cdot a_{ij}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\cdot x_i \cdot y_j$$

$$= X A Y^T \quad 3\times 3$$

$$\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{pmatrix}$$

$X = (\frac{1}{2}\ \frac{1}{4}\ \frac{1}{4})$

$Y = (\frac{1}{3}\ 0\ \frac{2}{3})$

$\frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot -1$

$\frac{1}{2}\cdot -1 + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 = -\frac{1}{4}$

$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot -1 + \frac{1}{4}\cdot 0$

$$\begin{pmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{3}\\ 0\\ \frac{2}{3}\end{pmatrix}$$

a best response to X is 2

a best response to Y is 1

$$= \begin{pmatrix}0 & -\frac{1}{4} & \frac{1}{4}\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{3}\\ 0\\ \frac{2}{3}\end{pmatrix} = \frac{1}{6}$$

$E(X,1)$, $E(X,2)$, $E(X,3)$

expected payoff for I if they play X and II plays col 1

	R	P	S
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

$E(1,Y) = \frac{1}{3}\cdot 0 + 0\cdot -1 + \frac{2}{3}\cdot 1 = \frac{2}{3}$

$E(2,Y) = -\frac{1}{3}$

$E(3,Y) = -\frac{1}{3}$

$E(X,Y) = \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{4}\cdot -\frac{1}{3} + \frac{1}{4}\cdot -\frac{1}{3}$

$= \frac{1}{6}$

A pair of mixed strategies X^*, Y^* is an equilibrium if

$$E(X,Y^*) \le E(X^*,Y^*) \le E(X^*,Y) \quad \text{for all } X, Y$$

II changes

I changes

Every game has an equilibrium in mixed strategies

Equilibrium Theorem: If X^*, Y^* is an equilibrium in mixed strategies for A with $x_i > 0$ $y_j > 0$ then

row i or col j in support of X^* or Y^*

$$E(X^*, j) = E(i, Y^*) = \text{value}(A) \leftarrow E(X^*, Y^*)$$

$$v(RPS) = 0 = E(X^*, 1) = E(X^*, 2) = E(X^*, 3)$$
$$= E^*(1, Y^*) = E(2, Y^*) = E(3, Y^*)$$

Value of a Game

Suppose there are two equilibria (x_1^*, y_1^*) (x_2^*, y_2^*)

I changing from equilibrium

$$E(x_1^*, y_2^*) \le \boxed{E(x_2^*, y_2^*)}$$ II changing from equil.

$$\le E(x_2^*, y_1^*)$$ I changing from equilibrium

$$\le E(x_1^*, y_1^*)$$ II changing from equilibrium

$$\le E(x_1^*, y_2^*)$$

Squeeze – all $\le$ are $=$

Best response : Best response to $\overset{\text{I}}{X}$ is a $\overset{\text{II}}{Y}$ that minimizes $E(X, Y)$

$\underset{\text{II}}{Y}$ is a $\underset{\text{I}}{X}$ that maxes $E(X, Y)$

Equilibrium strategies X^*, Y^* are best responses to each other

for any strategy X, there is a j that is a best response to X

Thm: X^*, Y^* is an equilibrium and value$(A) = E(X^*, Y^*)$

if and only if

$$E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j) \quad \text{for all } i, j$$

I changing from eq. II changing from equil.

$\Rightarrow$: from def of equilibrium

$\Leftarrow$: Suppose $E(i, Y^*) \le E(X^*, Y^*) \le E(X^*, j)$ for all i, j

[want X^*, Y^* is equilibrium: $\forall X, Y$ $E(X, Y^*) \le E(X^*, Y^*) \le E(X^*, Y)$]

similar

let $X = (x_1, \ldots, x_m)$

$$x_1 E(1, Y^*) \le E(X^*, Y^*) \cdot x_1$$
$$x_2 E(2, Y^*) \le E(X^*, Y^*) \cdot x_2$$
$$\vdots$$
$$x_m E(m, Y^*) \le E(X^*, Y^*) \cdot x_m$$

$$E(X, Y^*) \le E(X^*, Y^*) \cdot (x_1 + \cdots + x_m)$$
$$= E(X^*, Y^*)$$

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

Claim: $X^* = \left(\frac{2}{7}\ 0\ \frac{5}{7}\right)$ $Y^* = \left(\frac{5}{7}\ \frac{2}{7}\ 0\right)$ is an equilibrium

check $E(1, Y^*) \le E(X^*, Y^*)$ $\frac{2}{7}$ $\quad \frac{5}{7} \cdot 0.30 + \frac{2}{7} \cdot 0.25 \overset{?}{\le} \frac{2}{7}$ $\quad \frac{200}{700} \le \frac{2}{7}$ ✓

$E(2, Y^*) \le E(X^*, Y^*)$ $\quad \frac{5}{7} \cdot 0.26 + 0.33 \cdot \frac{2}{7} = \frac{196}{700} \overset{?}{\le} \frac{2}{7}$ ✓

$E(3, Y^*) \le E(X^*, Y^*)$ $\quad \frac{2}{7} \le \frac{2}{7}$ ✓

$E(X^*, 1) \ge E(X^*, Y^*)$ $\quad \frac{2}{7} \ge \frac{2}{7}$ ✓

$E(X^*, 2) \ge E(X^*, Y^*)$ $\quad \frac{2}{7} \ge \frac{2}{7}$ ✓

$E(X^*, 3) \ge E(X^*, Y^*)$ $\quad \frac{205}{700} \ge \frac{2}{7}$ ✓

all $\le$ $\ge$ worked, so X^*, Y^* is an equilibrium

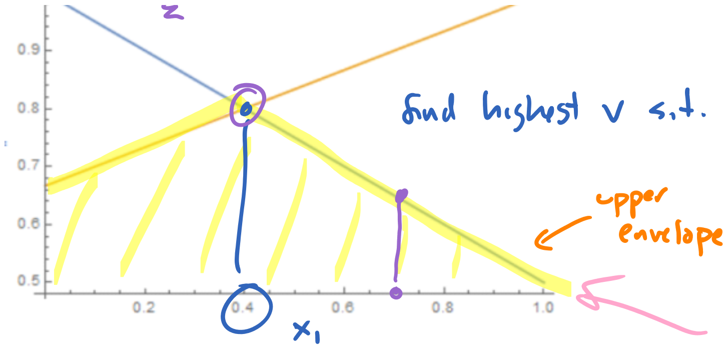
	L	R
L	$\frac{1}{2}$	1
R	1	$\frac{2}{3}$

want to find $X^* = (x_1\ \ 1 - x_1)$ s.t. $E(X^*, j) \ge E(X^*, Y^*)$ for $j = 1, 2$

$Y^* = (y_1\ \ 1 - y_1)$ $\quad E(i, Y^*) \le E(X^*, Y^*)$

$$E(X^*, 1) = x_1 \cdot \tfrac{1}{2} + (1 - x_1) \cdot 1$$
$$= 1 - \tfrac{1}{2} x_1 \ge v$$

$$E(X^*, 2) = x_1 + (1 - x_1) \cdot \tfrac{2}{3}$$
$$= \tfrac{2}{3} + \tfrac{1}{3} x_1 \ge v$$



find highest v s.t. $v \leq$ both $E(x^*,1)$ and $E(x^*,2)$

$$1 - \frac{1}{2}x_1 = \frac{2}{3} + \frac{1}{3}x_1$$

$$\frac{1}{3} = \frac{5}{6}x_1$$

$$\frac{2}{5} = x_1 \qquad x^* = \left(\frac{2}{5}\ \ \frac{3}{5}\right)$$

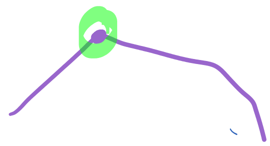
$$A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

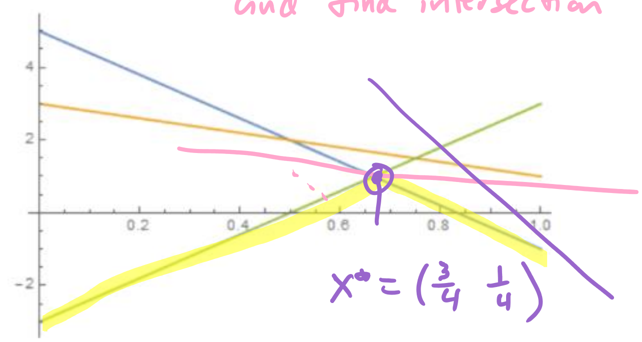
for II: graph $E(1,y^*)$, $E(2,y^*)$ and find intersection

$E(x,1) =$

$E(x,2) =$

$E(x,3) =$

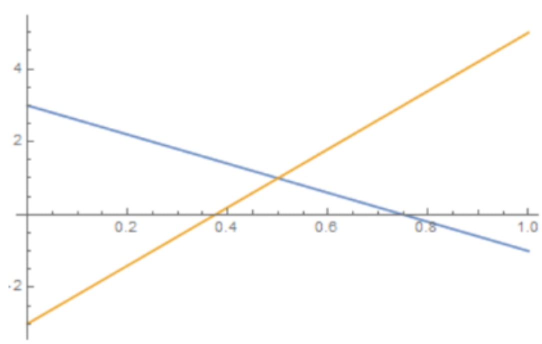




$x^* = \left(\frac{3}{4}\ \ \frac{1}{4}\right)$

$E(1,y) =$

$E(2,y) =$



$$A = \begin{pmatrix} 0 & -1 & 3 \\ 3 & 5 & -3 \end{pmatrix}$$

Linear Programming

	F	C	S
F	0.30	0.25	0.20
C	0.26	0.33	0.28
S	0.28	0.30	0.33

find $\underline{x} = (x_1 \ x_2 \ x_3)$ that satisfies

$$E(x,1) = 0.30\,x_1 + 0.26 \cdot x_2 + 0.28 \cdot x_3 \geq v$$

$$E(x,2) = 0.25\,x_1 + 0.33\,x_2 + 0.30\,x_3 \geq v$$

$$E(x,3) = 0.20\,x_1 + 0.28\,x_2 + 0.33\,x_3 \geq v$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

and maximizes v

linear programming (scipy.linprog)

divide by v to eliminate v

(after ensuring that $v > 0$ [by adding a c to A to make all entries > 0])

$$-0.30\frac{x_1}{v} - 0.26\frac{x_2}{v} - 0.28\frac{x_3}{v} \leq -1$$

$$-0.25\frac{x_1}{v} - 0.33\frac{x_2}{v} - 0.30\frac{x_3}{v} \leq -1$$

$$-0.20\frac{x_1}{v} - 0.28\frac{x_2}{v} - 0.33\frac{x_3}{v} \leq -1$$

$$\boxed{\text{let } p_i = \frac{x_i}{v}}$$

maximize v

$\Downarrow$

$$\boxed{\text{minimize } p_1 + p_2 + p_3} = \frac{x_1}{v} + \frac{x_2}{v} + \frac{x_3}{v} = \frac{(x_1 + x_2 + x_3)}{v} = \frac{1}{v}$$

c

$$-0.30\,p_1 - 0.26\,p_2 - 0.28\,p_3 \leq -1$$

$$-0.25\,p_1 - 0.33\,p_2 - 0.30\,p_3 \leq -1$$

$$-0.20\,p_1 - 0.28\,p_2 - 0.33\,p_3 \leq -1$$

b_ub

al

and $0 \leq p_i = \frac{x_i}{v} \leq \frac{1}{v} \leq 5$

bounds

since $x_i \leq 1$

because $v \geq 0.20$

lowest value in payoff matrix

result.x result.fun

LP returns (p_1, p_2, p_3) and $(p_1 + p_2 + p_3) = \frac{1}{v}$

recover x_i with $x_i = p_i \cdot v = \frac{p_i}{p_1 + p_2 + p_3}$

II

minimize v subject to

$$E(1,y) = 0.30\,y_1 + 0.25\,y_2 + 0.20\,y_3 \leq v$$

$$E(2,y) = 0.26\,y_1 + 0.33\,y_2 + 0.28\,y_3 \leq v$$

$$E(3, Y) = 0.28y_1 + 0.30y_2 + 0.33y_3 \le v$$

$\Downarrow$ divide by v, replace $\frac{y_i}{v}$ with q_i

$\begin{pmatrix} \min -\frac{1}{v} \\ \equiv \\ \max \frac{1}{v} \\ \equiv \\ \min v \end{pmatrix}$

find q_1, q_2, q_3 to minimize $-q_1 - q_2 - q_3 = -\frac{1}{v}$

$$0.30q_1 + 0.25q_2 + 0.20q_3 \le 1$$

$$0.26q_1 + 0.33q_2 + 0.28q_3 \le 1$$

$$0.28q_1 + 0.30q_2 + 0.33q_3 \le 1$$

$$0 \le q_i \le 5$$

LP returns q_i and $-q_1 - q_2 - q_3$

recover y_i with $y_i = q_i \cdot v = \frac{q_i}{q_1 + q_2 + q_3}$