1 Linear Systems, Superposition, and Convolution

In this section we provide a brief introduction to linear systems theory. It is based on the following definition:

Let $\mathbf{L}$ denote a *time-invariant, linear system* with input x and output y:

$$x \longrightarrow \boxed{\;\mathbf{L}\;} \longrightarrow y$$

Such systems are *additive* (the response to the sum of two inputs x_1 and x_2 equals the sum of the responses to each input taken individually):

$$\mathbf{L}(x_1 + x_2) = \mathbf{L}(x_1) + \mathbf{L}(x_2)$$

and *homogeneous* (the system can be scaled by the magnitude of the input α, where α is a scalar):

$$\mathbf{L}(\alpha x) = \alpha \mathbf{L}(x)$$

Taken together, they enjoy the *Principle of Superposition:*

$$\mathbf{L}(\alpha x_1 + \beta x_2) = \alpha \mathbf{L}(x_1) + \beta \mathbf{L}(x_2)$$

Several idealized input functions are of special importance in analyzing systems, the *Dirac delta function* and the *unit step function*. The unit step is defined as:

$$u(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t > 0; \\ \text{undefined}, & \text{if } t = 0 \end{cases} \tag{1}$$

and the Dirac delta function is its derivative:

$$\delta(t) \triangleq \frac{du(t)}{dt}.$$

Of course, since the step function is not continuous (notice the value at 0), one has to be careful in defining exactly what is meant by the above equation. Formally, the $\delta(t)$ is a *distribution* defined by the integral equation:

$$\int_{-\infty}^{\infty} v(t)\delta(t)dt = v(0).$$

It can be represented as the limit of a sequence of functions such as:

$$\delta_\epsilon(t) = \frac{1}{\pi t} \sin \frac{\pi t}{\epsilon}$$

with $\epsilon \rightarrow 0$, see Fig. 1; During this limiting process we have something that resembles a physical approximation to the unit impulse, the first test pattern used to evaluate the lateral inhibitory network. The result is called the *impulse response*

$$h(t) = \mathbf{L}[\delta(t)]$$

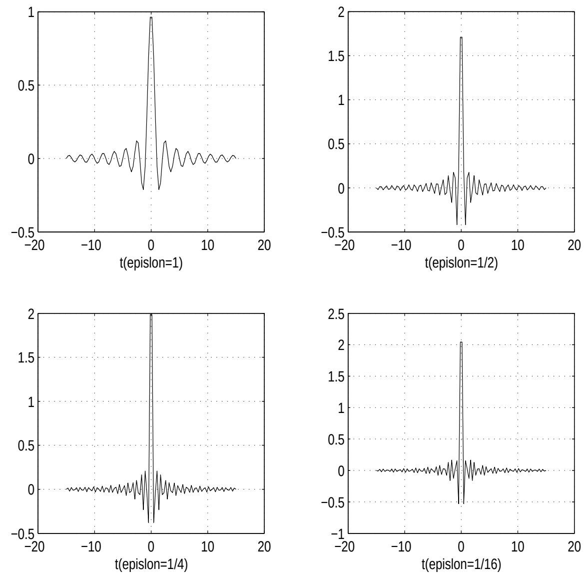

Figure 1:

Similarly, since the step function is the integral of the impulse function, the *step response*, or the response to a unit step function

$$S(t) = \mathbf{L}(u(t))$$

is the integral of the impulse response function:

$$S(t) = \int_{-\infty}^{t} h(\lambda) d\lambda.$$

In general, of course, we are interested in the response of a system not to these special functions, but to an arbitrary input. The trick is to approximate this arbitrary input as a sequence of step functions (Fig. 2.12 in Oppenheim and Willsky, p91), and then to use superposition to obtain the overall response. It is necessary to *assume that the input function* $x(t)$ *is continuous*, so that the approximation is meaningful. Now, suppose the input starts at $t = 0$, and time is discretized into bins Δ seconds apart. The first step in the approximation has height $x(0)$, a constant, and the additional step at $t = k\Delta$ has height $[x(k\Delta - x((k-1)\Delta]$. Remembering that $u(0) = 1$, the signal $x(t)$ can thus be approximated by

$$\begin{aligned} \hat{x}(t) &= x(0)u(t) + [x(\Delta) - x(0)]u(t - \Delta) + [x(2\Delta) - x(\Delta)]u(t - 2\Delta) + \cdots && (2) \\ &= x(0)u(t) + \sum_{k=1}^{\infty} \{x(k\Delta) - x((k-1)\Delta)\}u(t - k\Delta) && (3) \end{aligned}$$

The approximate output is then given by

$$\begin{aligned}
\hat{y}(t) &= x(0)S(t) + \sum_{k=1}^{\infty}\{x(k\Delta) - x((k-1)\Delta)\}S(t-k\Delta) &(4)\\
&= x(0)S(t) + \sum_{k=1}^{\infty}\{\frac{x(k\Delta) - x((k-1)\Delta)}{\Delta}\}S(t-k\Delta)\Delta &(5)
\end{aligned}$$

Now, introduce a limiting process in which the time steps become vanishingly close, so that $\Delta \to d\tau$, $k\Delta \to \tau$, the sum becomes an integral and we have:

$$\begin{aligned}
\lim_{\Delta\to d\tau} \hat{y}(t) &\to y(t) &(6)\\
&= x(0)S(t) + \int_{0+}^{\infty} \frac{dx(\tau)}{d\tau} S(t-\tau)d\tau &(7)
\end{aligned}$$

Repeating this approximation procedure for a continuous input $x(t)$ starting at $t' > -\infty$ and such that $x(t')S(t-t') \to 0$ as $t' \to -\infty$ yields:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} \frac{dx(\tau)}{d\tau} S(t-\tau)d\tau &(8)\\
&= \int_{-\infty}^{\infty} x(\tau)\frac{dS(\tau)}{d\tau}d\tau &(9)\\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &(10)
\end{aligned}$$

Thus the output can be computed as an integral function of the impulse response. Such an integral is called a *convolution integral*, and the above expression is often abbreviated:

$$y(t) = x(t) * h(t).$$

Reference

Alan V. Oppenheim and Alan S. Willsky, *Signals and Systems*, second edition, Prentice-Hall Inc., 1997.