Why are they difficult? → seemingly no information

Vandermonde's Identity
\[
\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}
\]

1. Focus on the easy side.
   Find a situation to describe it. (I like to do committees).
   Committee of \( r \) people from \( m+n \) people.

2. Think of a different way to model the situation
   \( \sum \rightarrow \text{cases} \). If stuck, focus on the end-values.
   Divide \( m+n \) into a group of \( m \) people and a group of \( n \) people

\[
\frac{m}{\circ} \quad \frac{n}{\circ}
\]

If \( 0 \) people chosen from group \( \circ \) \( m \) people, then all
remaining \( r \) people must be chosen from group \( \circ \) \( n \) people.

\[
\frac{m}{\circ} \quad \frac{n}{\circ} \rightarrow \binom{m}{0} \binom{n}{r}
\]

Similarly, \( \frac{m}{\circ} \quad \frac{n-1}{\circ} \rightarrow \binom{m}{1} \binom{n}{r-1} \)

\[
\vdots
\]

\[
\frac{m}{\circ} \quad 0 \rightarrow \binom{m}{r} \binom{n}{0}
\]

\[
\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}
\]

3. Make sure there is a bijection between the two ways of counting.

Hockey stick Identity
\[
\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}
\]

\[
\text{Committee of } r+1 \text{ people from } n+1 \text{ people.}
\]
Take some person, \( \frac{n!}{r! (n-r)!} \) cases

Case 2. This person is not in the committee

Case 2.1 Take another person and assume in committee

\[ \Rightarrow \binom{n-1}{r} \]

\[ \cdots \]

\[ \binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r} \]

Counting.

A good mathematician minimizes the number of cases.

Cases can quickly get out of hand.

Use every single piece of information!

HW 9 Problem #1 can be solved with O coursework.

Experience with non-textbook, complex problem brings respect for "clean" numbers vs. "dirty" numbers.

<table>
<thead>
<tr>
<th>Clean</th>
<th>Dirty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squares</td>
<td>Cubes</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>27</td>
</tr>
<tr>
<td>16</td>
<td>64</td>
</tr>
<tr>
<td>25</td>
<td>125</td>
</tr>
<tr>
<td>36</td>
<td>216</td>
</tr>
</tbody>
</table>

A: Base 6 representation of first = 216 non-negative integers.

No occurrence of 3

b) \( 5 + 20 + 100 = 125 \) !!!!

At least one occurrence of 1 or 4

c) Complementary counting: \( 4 + 12 + 48 = 64 \) !!!!!!! \( 216 - 64 = 152 \)
How can the numbers be so clean?
The number站立ed out clean! \( \frac{2}{16} \)

Clean numbers \( \Rightarrow \) Symmetry

\[
\begin{array}{c|c|c|c|c}
6 & 6 & 6 & \rightarrow & 216 \\
\hline
5 & 5 & 5 & \rightarrow & 125 \\
\end{array}
\]

Treat 1-digit & 2-digit #s as 3-digit #s with leading zeroes.

Only restriction: No occurrence of 3 \( \rightarrow 3^3 = 125 \)

\[
\begin{array}{c|c|c}
124 & \rightarrow & 4^3 = 64 \\
216 - 64 = 152 \\
\end{array}
\]

a) Symmetry allows isolation of digits.

For every digit, there are an equal # of other digits that make up the base-6 #.

How many occurrences of 4?

\[
\begin{array}{c|c|c|c|c|c|c}
4 & 1 & - & - & - & - & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \end{array}
\]

b) \( \frac{216}{6} \times \frac{216}{6} \times 3 = 108 \)

c) Added zeroes \( \Rightarrow \) should not be counted

\[
\begin{array}{c|c|c|c|c|c|c}
36 & - & 0 & 0 & - & 108 - 36 - 6 = 66 \\
\end{array}
\]

Exploited Symmetry

Ball-and-Urn

7 fair 6-sided dice are thrown. The probability that the sum of the #s on the top face is 10 is of the form \( \frac{n}{67} \).

Divide 10 into 7 positive #s.
In tens, 7 received. 6 dividers.

\[
\binom{10-7+6}{6} = \binom{9}{6} = \frac{9!}{6!3!} = 84
\]

\[+\]

\[
\binom{9}{6} = 84
\]

Number of 7-digit positive integers that have the property that their digits are in nondecreasing order. Ex) 1234567, 11223549

Observation: 0 is not allowed. Once we know the digits involved, we know their placement, but not the # of each.

Solution I. Covexnum:

7 digits chosen: \( \binom{9}{7} \times \binom{6}{2} \)

6

\[
\binom{9}{6} \times \binom{6}{1}
\]

5

\[
\binom{9}{5} \times \binom{7}{2}
\]

4

\[
\binom{9}{4} \times \binom{8}{2}
\]

3

\[
\binom{9}{3} \times \binom{9}{2}
\]

1

\[
\binom{9}{1} \times \binom{10}{2}
\]

\[
\binom{9}{7} \times \binom{6}{2} + \binom{9}{6} \times \binom{6}{1} + \ldots + \binom{9}{1} \times \binom{10}{2}
\]

\[
= \binom{15}{7} = 6435
\]

Solution II.

Treat like the unknown digits have count of 8.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \rightarrow \text{digit positions} \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast
\end{array}
\]

digits \rightarrow 8 dividers. 1244667 \( \binom{7+8}{8} = \binom{15}{8} = \binom{15}{7} = 6435 \)