Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. (You might check out Latex if you are not familiar with it.) Include your name, your netid, the homework number and the problem number on EACH PART.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 2 points on this homework.

Partial credit will be given if and only if the grader can easily understand enough of your answer to award it.

1. (24 points) Let \( A, B, \) and \( C \) be unary predicates. For each of the following “maybe syllogisms”, determine if the third line is a logical consequence of the first two. If so, give an informal proof to show that it is a logical consequence. If not, give an example of a universe (set of elements) and definitions of the predicates \( A, B, \) and \( C \) for which the first two lines are true but the third is false.

(a) i. \((\forall x)(A(x) \rightarrow B(x))\)
ii. \((\forall x)(A(x) \land B(x) \rightarrow C(x))\)
iii. \((\forall x)(A(x) \rightarrow C(x))\)

Solutions. Statement (iii) is a logical consequence of statements (i) and (ii).

Proof. Assume that (i) and (ii) are true in some domain. Let \( a \) be an arbitrary element of the domain. Assume \( A(a) \) is true. Then, by (i), \( B(a) \) is true, so \( A(a) \land B(a) \) is also true. Thus, by (ii), \( C(a) \) is true, and \( A(a) \rightarrow C(a) \) is true. Since \( a \) was arbitrary, 
\((\forall x)(A(x) \rightarrow C(x))\) is true.

(b) i. \((\exists x)(\neg A(x) \lor \neg C(x))\)
ii. \((\forall x)(B(x) \rightarrow C(x))\)
iii. \((\exists x)(\neg A(x) \land \neg B(x))\)

Solutions. The statement (iii) is a not logical consequence of the statements (i) and (ii).

To see this, consider a domain with one element \( a \), where \( A(a) = 1, B(a) = 0, \) and \( C(a) = 0 \). Then (i) is true in this domain because \( \neg C(a) \) is true, so \( (\neg A(a) \lor \neg C(a)) \) is true, and \((\exists x)(\neg A(x) \lor \neg C(x))\) is true. Also, (ii) is true in this domain because (considering the only element, \( a \), of the domain), \( B(a) = 0, \) so \( B(a) \rightarrow C(a) \) is true, and \((\forall x)(B(x) \rightarrow C(x))\) is true. However, (iii) is false in this domain, because \( \neg A(a) \) is false, so \( \neg A(a) \land \neg B(a) \) is false, and because \( a \) is the only element of the domain, 
\((\exists x)(\neg A(x) \land \neg B(x))\) is false.

(c) i. \((\forall x)(A(x) \rightarrow B(x))\)
ii. \((\forall x)(C(x) \rightarrow \neg B(x))\)
iii. $(\forall x)(\neg A(x) \lor \neg C(x))$

**Solution.** Statement (iii) follows logically from statements (i) and (ii).

Proof: Assume (i) and (ii) are true in some domain. Let $a$ be an arbitrary element of the domain. We'll show by contradiction that $(\neg A(a) \lor \neg C(a))$ must be true. If not, then it must be that $A(a)$ is true, which by (i) implies $B(a)$ is true, and also $C(a)$ must be true, which by (ii) implies $\neg B(a)$ is true, which is a contradiction. Thus, $(\neg A(a) \lor \neg C(a))$ must be true. Since $a$ was arbitrary, we conclude that $(\forall x)(\neg A(x) \lor \neg C(x))$ is true.

(d) i. $(\exists x)(A(x) \land \neg B(x))$
ii. $(\forall x)(\neg C(x) \rightarrow B(x))$
iii. $(\exists x)(A(x) \land C(x))$

**Solution.** Statement (iii) follows logically from statements (i) and (ii).

Proof: Assume that (i) and (ii) are true in some domain. By (i), there exists an element $a$ such that $(A(a) \land \neg B(a))$ is true. Thus, $A(a)$ is true and $\neg B(a)$ is true. By (ii), we know $(\neg C(a) \rightarrow B(a))$ is true, and therefore $(\neg B(a) \rightarrow C(a))$ (which is logically equivalent to it) is also true. Since $\neg B(a)$ is true, we conclude that $C(a)$ is true, and therefore, $(A(a) \land C(a))$ is true. Since there is at least one element $a$ with this property, $(\exists x)(A(x) \land C(x))$ is true.

2. (25 points) In this problem, assume that the domain is all nonnegative integers, so a “number” refers to one of $0, 1, 2, \ldots$. The statement language includes the usual predicate logic symbols together with the constants 0 and 1 (for the numbers zero and one), the function symbols + and · (for the operations of plus and times), and the predicate symbols $=, \neq, <, \leq, \geq,$ and $>$ (for equal, not equal, less than, less than or equal to, greater than or equal to, and greater than, respectively.)

For each of the mathematical statements below, write closed predicate logic formulas (using only the given symbols) for the statement and its negation. **Annoying Restriction Just For Problem (2):** the symbol $\neg$ (or its equivalent) must not appear in the statement or in its negation. Identify which of the two statements is true in this domain, and give an informal proof of your answer.

Note that the square of a number $x$ is $x \cdot x$. Specifications like “two numbers” or “three numbers” do not require that the numbers be different.

As an example, assume the statement is “For every number, there is a number that is not equal to it.” A suitable answer would be:

Statement: $(\forall x)(\exists y)(x \neq y)$

Negation: $(\exists x)(\forall y)(x = y)$

The statement is true, because given any number $x$, the number $y = x + 1$ is not equal to $x$. (We don’t have to consider its negation separately, because its negation must therefore be false.)

(a) The sum of any two numbers is greater than either of them.

**Solution.**

Statement: $(\forall x)(\forall y)((x + y > x) \land (x + y > y))$
Negation: $(\exists x)(\exists y)((x + y \leq x) \lor (x + y \leq y))$

The negation is true, because we can take $x = 0$ and $y = 0$, and then $x + y = 0$, so $x + y \leq x$ (and also $x + y \leq y$).

(b) The product of any two numbers that are greater than zero is greater than zero

Solution.

Statement: $(\forall x)(\forall y)((x > 0) \land (y > 0) \rightarrow (x \cdot y > 0))$

Negation: $(\exists x)(\exists y)((x > 0) \land (y > 0) \land (x \cdot y \leq 0))$

The statement is true, because for any nonnegative integers $x$ and $y$, the only way $x \cdot y$ is not greater than 0 is if $x \cdot y = 0$, in which case we have $x = 0$ or $y = 0$, that is, not both $x > 0$ and $y > 0$.

(c) For any number there exists a number smaller than it.

Solution.

Statement: $(\forall x)(\exists y)(y < x)$

Negation: $(\exists x)(\forall y)(y \geq x)$

The negation is true, because we may take $x = 0$, and then for any nonnegative integer $y$, $y \geq x$.

(d) For any number, the square of the next number is greater than the original number.

Solution.

Statement: $(\forall x)((x + 1) \cdot (x + 1) > x)$

Negation: $(\exists x)((x + 1) \cdot (x + 1) \leq x)$

The statement is true. Let $x$ be any nonnegative integer. Then $(x + 1) \cdot (x + 1) = x^2 + 2x + 1 \geq x + 1 > x$.

(e) Solution.

There exist three nonzero numbers such that the sum of their squares is also a square number.

Statement: $(\exists x)(\exists y)(\exists z)(((x \neq 0) \land (y \neq 0) \land (z \neq 0) \land (x \cdot x + y \cdot y + z \cdot z = w \cdot w))$

Negation: $(\forall x)(\forall y)(\forall z)(((x = 0) \lor (y = 0) \lor (z = 0) \lor (x \cdot x + y \cdot y + z \cdot z \neq w \cdot w)))$

The statement is true, since we could take $x = 1$, $y = 2$, $z = 2$ and $w = 3$, for which we have $1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 = 9 = 3 \cdot 3$.

3. (25 points) For this problem, assume the domain and symbols specified in problem (2), as well as the constant symbol 2 (for the number two) and unary predicates even($x$) and odd($x$), where even($x$) is true if and only if $(\exists y)(x = 2 \cdot y)$ is true, and odd($x$) is true if and only if $(\exists y)(x = (2 \cdot y) + 1)$ is true.

Each of the following statements is true in this domain. Translate each statement into a predicate logic formula using the given symbols, and give an informal proof of the statement. Note that some of these statements have implied “for all” quantifiers, which you should make explicit; this applies also to problem (4).

(a) A number is even if and only if it is not odd. (Hint: consider dividing by 2.)

Solution.
Statement: \((\forall x)(\text{even}(x) \leftrightarrow \neg \text{odd}(x))\)
Proof: If \(x\) is any nonnegative integer, then the remainder of \(x\) divided by 2 is either 0 (in which case \(x\) is even) or 1 (in which case \(x\) is odd), but not both. Thus, if \(x\) is even, it is not odd, and if \(x\) is not odd then \(x\) is even.

(b) The sum of an even number and an odd number is odd.
Solution.

Statement: \((\forall x)(\forall y)(\text{even}(x) \land \text{odd}(y) \rightarrow \text{odd}(x + y))\)
Proof: Let \(x\) and \(y\) be arbitrary nonnegative integers and assume that \(x\) is even and \(y\) is odd. Then there exist numbers \(w\) and \(z\) such that \(x = 2w + 1\) and \(y = 2z + 1\). Thus, \(x + y = 2(w + z) + 1\), so \(x + y\) is odd. Since \(x\) and \(y\) were arbitrary, the statement follows.

(c) The product of two odd numbers is odd.
Solution.

Statement: \((\forall x)(\forall y)(\text{odd}(x) \land \text{odd}(y) \rightarrow \text{odd}(x \cdot y))\)
Proof: Let \(x\) and \(y\) be arbitrary nonnegative integers and assume that they are both odd. Then there exist numbers \(w\) and \(z\) such that \(x = 2w + 1\) and \(y = 2z + 1\). Thus, \(xy = 4wy + 2(w + z) + 1\), so \(xy\) is odd. Since \(x\) and \(y\) were arbitrary, the statement follows.

(d) A number is even if and only if its square is even.
Solution.

Statement: \((\forall x)(\text{even}(x) \leftrightarrow \text{even}(x 
\cdot x))\)
Proof: Let \(x\) be an arbitrary nonnegative integer. If \(x\) is even, then there exists a number \(w\) such that \(x = 2w\), so \(x^2 = 4w^2\) and \(x^2\) is even. For the converse, suppose that \(x^2\) is even. If \(x\) is not even, then by (a) \(x\) is odd, and by (c), \(x^2\) is odd, which by (a) implies \(x^2\) is not even, a contradiction. Thus \(x\) must be even, so \(x\) is even if and only if \(x^2\) is even. Since \(x\) was arbitrary, the statement follows.

(e) The sum of a number and its square is even.
Solution.

Statement: \((\forall x)(\text{even}(x + x \cdot x))\)
Proof: We first observe that the sum of two even numbers is even (because \(2w + 2z = 2(w + z)\)) and the sum of two odd numbers is even (because \(2w + 1 + 2z + 1 = 2(w + z + 1)\)). Let \(x\) be an arbitrary nonnegative integer. We argue by cases: (1) if \(x\) is even, then by (d) \(x^2\) is even, and the sum of two even numbers is even, so \(x + x^2\) is even. For case (2), \(x\) is odd, so by (c) \(x^2\) is odd, and the sum of two odd numbers is even, so \(x + x^2\) is even in this case as well. Since \(x\) was arbitrary, the statement follows.

4. (24 points) For this problem, assume the domain and symbols of problem (3), and also the constant symbol 4 (for the number four) and the binary predicate symbol divides\((x, y)\) (where divides\((x, y)\) is true if and only if \((\exists z)(y = z \cdot x)\) is true).

The statements (a)-(c) below are true in this domain. For each of these statements, translate it into a closed predicate logic formula using the given symbols, and give an informal proof of the statement. For (d), answer the given question.

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(a) If $x$ divides both $y$ and $z$ then it divides both their sum and their product.

\textit{Solution.}

\textit{Statement:}

$$(\forall x)(\forall y)(\forall z)(\text{divides}(x, y) \land \text{divides}(x, z) \rightarrow \text{divides}(x, y + z) \land \text{divides}(x, y \cdot z))$$

\textit{Proof:} Let $x$, $y$, and $z$ be arbitrary nonnegative integers and assume that $x$ divides $y$ and $x$ divides $z$. Then there exist numbers $d_1$ and $d_2$ such that $y = d_1 x$ and $z = d_2 x$. Thus, $y + z = (d_1 + d_2)x$, so $x$ divides $y + z$, and $yz = (d_1)(d_2)x^2$, so $x$ divides $yz$. Since $x$, $y$, and $z$ were arbitrary, the statement follows.

(b) There is no number $x$ such that $4$ divides both $x$ and $x + 2$.

\textit{Solution.}

\textit{Statement:} $\neg(\exists x)(\text{divides}(4, x) \land \text{divides}(4, x + 2))$

\textit{Proof:} Suppose to the contrary that $x$ is a nonnegative integer such that $4$ divides both $x$ and $x + 2$. Then there exist numbers $w$ and $z$ such that $x = 4w$ and $x + 2 = 4z$. Thus, $2 = (x + 2) - x = 4(z - w)$, which is a contradiction, because $4$ does not divide $2$. Thus, there can exist no such nonnegative integer $x$.

(c) If $x$ and $y$ are not both even, then the square of $x$ is not equal to twice the square of $y$.

\textit{Solution.}

\textit{Statement:} $$(\forall x)(\forall y)(\neg(\text{even}(x) \land \text{even}(y)) \rightarrow (x \cdot x \neq 2 \cdot (y \cdot y)))$$

\textit{Proof:} Let $x$ and $y$ be arbitrary nonnegative integers. Assume that $x$ and $y$ are not both even. Then by 3(a), at least one of them is odd. We consider cases: (1) Assume $x$ is odd. Then by 3(c), $x^2$ is odd. Clearly, $2y^2$ is even, so it must be that $x^2 \neq 2y^2$ because $x^2$ cannot be both even and odd. For the second case, assume $x$ is even and $y$ is odd. Then there exist numbers $w$ and $z$ such that $x = 2w$ and $y = 2z + 1$. We have $x^2 = 4w^2$, so $4$ divides $x^2$. Also,

$$2y^2 = 2(4z^2 + 4z + 1) = 8z^2 + 8z + 2 = 4(2z^2 + 2z) + 2.$$  

Since $4$ divides $4(2z^2 + 2z)$, by (b), $4$ does not divide $2y^2$. Because $4$ divides $x^2$ and does not divide $2y^2$, $x^2 \neq 2y^2$ in this case also. Because this holds in both cases, and $x$ and $y$ were arbitrary, the statement follows.

(d) Explain how the statement in (c) implies that the square root of two is irrational. (You may have to look up the definitions of rational and irrational numbers.)

\textit{Solution.}

We prove this by contradiction. Suppose that the square root of $2$ is rational. Then there exist nonnegative integers $p$ and $q$ such that $q \neq 0$ and

$$\frac{p}{q} = \sqrt{2}.$$  

We may assume that the fraction $p/q$ is in lowest terms, that is, the greatest common divisor of $p$ and $q$ is $1$. Squaring both sides of the equation, we get

$$\frac{p^2}{q^2} = 2,$$  

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which implies that $p^2 = 2q^2$. Since $p$ and $q$ cannot both be even (otherwise, their greatest common divisor would be at least 2), we note that this contradicts the result in part (c). Hence the square root of 2 is irrational.