Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. (You might check out Latex if you are not familiar with it.) Include your name, your netid, the homework number and the problem number on EACH PART.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 2 points on this homework.

Partial credit will be given if and only if the grader can easily understand enough of your answer to award it.

1. (24 points) Let $A$, $B$, and $C$ be unary predicates. For each of the following “maybe syllogisms”, determine if the third line is a logical consequence of the first two. If so, give an informal proof to show that it is a logical consequence. If not, give an example of a universe (set of elements) and definitions of the predicates $A$, $B$, and $C$ for which the first two lines are true but the third is false.

(a) i. $(\forall x)(A(x) \rightarrow B(x))$
   ii. $(\forall x)(A(x) \land B(x) \rightarrow C(x))$
   iii. $(\forall x)(A(x) \rightarrow C(x))$

(b) i. $(\exists x)(\neg A(x) \lor \neg C(x))$
   ii. $(\forall x)(B(x) \rightarrow C(x))$
   iii. $(\exists x)(\neg A(x) \land \neg B(x))$

(c) i. $(\forall x)(A(x) \rightarrow B(x))$
   ii. $(\forall x)(C(x) \rightarrow \neg B(x))$
   iii. $(\forall x)(\neg A(x) \lor \neg C(x))$

(d) i. $(\exists x)(A(x) \land \neg B(x))$
   ii. $(\forall x)(\neg C(x) \rightarrow B(x))$
   iii. $(\exists x)(A(x) \land C(x))$

2. (25 points) In this problem, assume that the domain is all nonnegative integers, so a “number” refers to one of $0, 1, 2, \ldots$. The statement language includes the usual predicate logic symbols together with the constants 0 and 1 (for the numbers zero and one), the function symbols $+$ and $\cdot$ (for the operations of plus and times), and the predicate symbols $=, \neq, <, \leq, \geq,$ and $>$ (for equal, not equal, less than, less than or equal to, greater than or equal to, and greater than, respectively.)

For each of the mathematical statements below, write closed predicate logic formulas (using only the given symbols) for the statement and its negation. **Annoying Restriction Just For Problem (2):** the symbol $\neg$ (or its equivalent) must not appear in the statement or in its negation. Identify which of the two statements is true in this domain, and give an informal proof of your answer.
Note that the square of a number \( x \) is \( x \cdot x \). Specifications like “two numbers” or “three numbers” do not require that the numbers be different.

As an example, assume the statement is “For every number, there is a number that is not equal to it.” A suitable answer would be:

- **Statement**: \((\forall x)(\exists y)(x \neq y)\)
- **Negation**: \((\exists x)(\forall y)(x = y)\)

The statement is true, because given any number \( x \), the number \( y = x + 1 \) is not equal to \( x \). (We don’t have to consider its negation separately, because its negation must therefore be false.)

(a) The sum of any two numbers is greater than either of them.
(b) The product of any two numbers that are greater than zero is greater than zero.
(c) For any number there exists a number smaller than it.
(d) For any number, the square of the next number is greater than the original number.
(e) There exist three nonzero numbers such that the sum of their squares is also a square number.

3. (25 points) For this problem, assume the domain and symbols specified in problem (2), as well as the constant symbol 2 (for the number two) and unary predicates \( \text{even}(x) \) and \( \text{odd}(x) \), where \( \text{even}(x) \) is true if and only if \((\exists y)(x = 2 \cdot y)\) is true, and \( \text{odd}(x) \) is true if and only if \((\exists y)(x = (2 \cdot y) + 1)\) is true.

Each of the following statements is true in this domain. Translate each statement into a predicate logic formula using the given symbols, and give an informal proof of the statement. Note that some of these statements have implied “for all” quantifiers, which you should make explicit; this applies also to problem (4).

(a) A number is even if and only if it is not odd. (Hint: consider dividing by 2.)
(b) The sum of an even number and an odd number is odd.
(c) The product of two odd numbers is odd.
(d) A number is even if and only if its square is even.
(e) The sum of a number and its square is even.

4. (24 points) For this problem, assume the domain and symbols of problem (3), and also the constant symbol 4 (for the number four) and the binary predicate symbol \( \text{divides}(x, y) \) (where \( \text{divides}(x, y) \) is true if and only if \((\exists z)(y = z \cdot x)\) is true).

The statements (a)-(c) below are true in this domain. For each of these statements, translate it into a closed predicate logic formula using the given symbols, and give an informal proof of the statement. For (d), answer the given question.

(a) If \( x \) divides both \( y \) and \( z \), then it divides both their sum and their product.
(b) There is no number \( x \) such that 4 divides both \( x \) and \( x + 2 \).
(c) If \( x \) and \( y \) are not both even, then the square of \( x \) is not equal to twice the square of \( y \).
(d) Explain how the statement in (c) implies that the square root of two is irrational. (You may have to look up the definitions of rational and irrational numbers.)