Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. (You might check out Latex if you are not familiar with it.) Include your name, your netid, the homework number and the problem number on EACH PART.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 1 point on this homework.

Partial credit will be given if and only if the grader can easily understand enough of your answer to award it.

The symbol \( \mathbb{N} \) denotes the nonnegative integers, that is, \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \). The symbol \( \mathbb{R} \) denotes the real numbers.

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1. (25 points) For each of the logical statements below, classify the statement as true or false, and give a proof of your answer. Each step of your proof must be justified by an axiom, lemma or theorem cited (by number or name) from Chapter 4 of the text.

(a) \( (\forall a \in \mathbb{R})(a \leq 0) \rightarrow (0 \leq -a) \)

\textit{Solution.} This is true. Let \( a \) be an arbitrary element of \( \mathbb{R} \) and assume \( a \leq 0 \). By additive inverse (4.1.5), \( -a \) is such that \( a + (-a) = 0 \). By translation invariance (4.2.4), we can add \( -a \) to both sides of \( a \leq 0 \) to get \( a + (-a) \leq 0 + (-a) \). Because \( -a \) is the additive inverse of \( a \), \( a + (-a) = 0 \), and because 0 is the additive identity (4.1.3), \( 0 + (-a) = (-a) \), so \( 0 \leq (-a) \). Since \( a \) was arbitrary, the statement is proved.

(b) \( (\forall x \in \mathbb{R})(0 \leq x) \rightarrow (x \leq x \cdot x) \)

\textit{Solution.} This is false. If we take \( x = 0.5 \), then \( 0 \leq x \) but \( x \cdot x = 0.25 \), and 0.5 is not less than or equal to 0.25, so \( x \) is not less than or equal to \( x \cdot x \).

(c) \( (\forall a, b, c, d \in \mathbb{R})(a \leq b) \land (c \leq d) \rightarrow ((a - d) \leq (b - c)) \)

\textit{Solution.} This is true. Let \( a, b, c, d \) be arbitrary elements of \( \mathbb{R} \) and assume that \( a \leq b \) and \( c \leq d \). By Theorem 4.2.11, \( (a+c) \leq (b+d) \). Applying additive inverse (4.1.5), \( c \) has additive inverse \( -c \) and \( d \) has additive inverse \( -d \). By translation invariance (4.2.4), we can add \( -c - d \) to both sides of \( (a+c) \leq (b+d) \) to get \( (a + c) + (c - d) \leq (b + d) + (d - c) \). Applying commutativity (4.1.1) and...
associativity (4.1.2) of $\cdot$, this implies $(a - d) + (c + (−c)) \leq (b - c) + (d + (−d))$, and by additive inverse (4.1.5) we have $(a - d) + 0 \leq (b - c) + 0$, and by additive identity (4.1.3), we have $(a - d) \leq (b - c)$. Because $a, b, c, d$ were arbitrary, the statement is proved.

(d) \(\forall a, b, c, d \in \mathbb{R} ((0 \leq a) \land (a \leq b) \land (c \leq d) \rightarrow (a \cdot c \leq b \cdot d))\).

Solution. This is false. Consider $a = 1, b = 2, c = −3, d = −2$. Then $0 \leq a$ and $a \leq b$ and $c \leq d$ but $a \cdot c = −3$ and $b \cdot d = −4$ and $a \cdot c < b \cdot d$.

(e) \(\forall a, b \in \mathbb{R} ((a < b) \rightarrow (\exists c \in \mathbb{R} ((a < c) \land (c < b))))\).

Solution. This is true. Let $a$ and $b$ be arbitrary real numbers. Assume $a < b$. By translation invariance, we may add $a$ to both sides of the inequality to get $a + a < b + a$, which by commutativity of $\cdot$ (4.1.1) implies $a + a < a + b$. Using the fact that 1 is an identity for multiplication (4.1.9), this implies $1 \cdot a + 1 \cdot a < a + b$, and by the distributive law (4.1.11), $(1 + 1) \cdot a < a + b$. The number $(1 + 1)$ (call it 2) is not 0 and has a multiplicative inverse (4.1.10) $2^{-1}$, which is not negative (4.1.15), so by scaling we get $2^{-1} \cdot 2a < 2^{-1} (a + b)$, or $a < 2^{-1} (a + b)$. A similar argument starting by adding $b$ to both sides of $a < b$ shows that $2^{-1} (a + b) < b$, so by choosing $c = 2^{-1} (a + b)$, we find a real number such that $a < c$ and $c < b$. The statement is proved.

In each of the proofs by induction in problems (2), (3), and (4), you must explicitly state and label the goal, the predicate $P(n)$, the base case(s), the proof of the base case(s), the statement of the inductive step, and its proof. Your proofs should have English sentences connecting and justifying the formulas. As an example of the specified format, consider the proof done in lecture Thursday (9/26).

**Goal:** To prove that for all $n \in \mathbb{N}$, if $n \geq 1$ then $1 + 2 + \ldots + n = n(n + 1)/2$.

**Predicate:** The predicate $P(n)$ is

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2}.$$ 

**Base Case:** The base case is $P(1)$.

**Proof:** $P(1)$ is true because $1 = (1 \cdot 2)/2$.

**Inductive step:** The statement of the inductive step is

$$(\forall n \in \mathbb{N}) ((n \geq 1) \land P(n) \rightarrow P(n + 1))$$.

**Proof:** Let $n$ be an arbitrary element of $\mathbb{N}$. Assume that $n \geq 1$ and $P(n)$ are true. Then we know

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2}.$$
Adding \((n+1)\) to both sides of this equation, we have

\[
1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1)
\]

\[
= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}
\]

\[
= \frac{(n+1)(n+2)}{2}
\]

The last statement shows that \(P(n+1)\) is true, so the inductive step is proved.

2. (25 points) Prove by simple induction, using the specified format, that for all \(n \in \mathbb{N}\), if \(n \geq 1\) then the sum of the first \(n\) odd positive integers is \(n^2\).

Solution.

**Goal:** To prove that for all \(n \in \mathbb{N}\), if \(n \geq 1\), then \(1 + 3 + 5 + \ldots + (2n-1) = n^2\).

**Predicate:** The predicate \(P(n)\) is

\[1 + 3 + 5 + \ldots + (2n-1) = n^2.\]

**Base case:** The base case is \(P(1)\).

**Proof:** \(P(1)\) is true because \((2 \cdot 1 - 1) = 1^2\).

**Inductive Step:** The statement of the inductive step is

\[(\forall n \in \mathbb{N})(\,(n \geq 1) \land P(n) \rightarrow P(n+1) ).\]

**Proof:** Let \(n \in \mathbb{N}\) be arbitrary. Assume \(n \geq 1\) and \(P(n)\) are true. This implies that

\[1 + 3 + 5 + \ldots + (2n-1) = n^2.\]

Adding the next odd number, \((2n+1)\) to both sides of this equality, we have

\[1 + 3 + 5 + \ldots + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2,\]

which shows that \(P(n+1)\) is true, so the inductive step is proved.

3. (25 points) Prove by simple induction, using the specified format, that for all \(n \in \mathbb{N}\), if \(n \geq 3\) then \(2n^2 \geq 5n + 2\).

Solution.

**Goal:** To prove that for all \(n \in \mathbb{N}\), if \(n \geq 3\) then \(2n^2 \geq 5n + 2\).

**Predicate:** The predicate \(P(n)\) is

\[2n^2 \geq 5n + 2.\]
Base case: The base case is $P(3)$.

Proof: $P(3)$ is true because $2(3)^2 = 18 \geq 17 = 5 \cdot 3 + 2$.

Inductive step: $(\forall n \in \mathbb{N})(n \geq 3 \land P(n) \rightarrow P(n + 1))$.

Proof: Let $n \in \mathbb{N}$ be arbitrary. Assume $n \geq 3$ and $P(n)$ are true. Then we have that

$$2n^2 \geq 5n + 2.$$ 

We add $4n + 2$ to both sides of this equality, which gives

$$2n^2 + 4n + 2 \geq 5n + 2 + 4n + 2,$$

which is equivalent to

$$2(n + 1)^2 \geq 9n + 4.$$ 

However, because $n \geq 3$, we have $4n \geq 12$. Thus, $9n + 4 \geq 5n + 16$, and $5n + 16 > 5n + 7$, so by transitivity,

$$2(n + 1)^2 > 5n + 7 = 5(n + 1) + 2,$$

and $P(n + 1)$ is true. This proves the inductive step.

4. (25 points) Prove by strong induction, using the specified format, that for all $n \in \mathbb{N}$, if $n \geq 8$ then there exist nonnegative integers $a$ and $b$ such that $n = 3a + 5b$.

Solution.

Goal: To prove that for all $n \in \mathbb{N}$, if $n \geq 8$ then there exist $a, b \in \mathbb{N}$ such that $n = 3a + 5b$.

Predicate: The predicate $P(n)$ is

$$(\exists a, b \in \mathbb{N})(n = 3a + 5b).$$

Base cases: The base cases are $P(8)$, $P(9)$, and $P(10)$.

Proof: $P(8)$ is true because $8 = 3 \cdot 1 + 5 \cdot 1$. $P(9)$ is true because $9 = 3 \cdot 3 + 5 \cdot 1$. $P(10)$ is true because $10 = 3 \cdot 0 + 5 \cdot 2$.

Inductive step: The statement of the inductive step is

$$(\forall n \in \mathbb{N})(n \geq 10) \land ((\forall m \in \mathbb{N})(m \geq 8) \land (m \leq n) \rightarrow P(m)) \rightarrow P(n + 1)).$$

Proof: Let $n \in \mathbb{N}$ be arbitrary. Assume $n \geq 10$ and that for all $m \in \mathbb{N}$ such that $8 \leq m \leq n$, we have $P(m)$ is true. Note that if $m = n - 2$, then $m$ is greater than or equal to 8 and less than or equal to $n$. Therefore, by our assumption,
$P(n - 2) = P(m)$ is true, that is, there exist nonnegative integers $c$ and $d$ such that

$$n - 2 = 3c + 5d.$$  

Adding 3 to both sides of this equality,

$$n + 1 = 3(c + 1) + 5d.$$  

Hence, taking $a = c + 1$ and $b = d$, we have that $n + 1 = 3a + 5b$, that is, $P(n+1)$ is true, so the inductive step is proved.