Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. (You might check out Latex if you are not familiar with it.) Include your name, your netid, the homework number and the problem number on EACH PART.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 1 point on this homework.

Partial credit will be given if and only if the grader can easily understand enough of your answer to award it.

For this assignment, while you may consult outside resources, any facts that you use in your proofs must be cited from the course text, Notes on Discrete Math, rather than outside resources.

1. (a) (10 points) Prove that for all positive integers $n$

$$
\sum_{i=1}^{n} (2i + 3) = n^2 + 4n.
$$

Solution. By the linearity of summation (Lemma 6.1.2),

$$
\sum_{i=1}^{n} (2i + 3) = 2 \sum_{i=1}^{n} i + 3 \sum_{i=1}^{n} 1.
$$

Using two of the three “standard sums” from Section 6.4.1,

$$
\sum_{i=1}^{n} (2i + 3) = 2 \frac{n(n+1)}{2} + 3n,
$$

and thus,

$$
\sum_{i=1}^{n} (2i + 3) = n^2 + n + 3n = n^2 + 4n.
$$

(This could also be proved directly using mathematical induction)
For parts (b) and (c), we define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sum_{i=0}^{n} (-1)^i(2i).$$

(b) (5 points) Make a table of the values of $f(n)$ for $n = 0, 1, 2, 3, 4$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(n)$</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>-4</td>
<td>4</td>
</tr>
</tbody>
</table>

(c) (10 points) Prove that for all natural numbers $n$, if $n$ is even then $f(n) = n$.

(Hint: you might want to prove something stronger than this.)

Solution. We prove that if $n$ is even then $f(n) = n$ and if $n$ is odd then $f(n) = -(n+1)$ by simple induction on the natural numbers. The predicate $P(n)$ is that $f(n) = n$ if $n$ is even and $f(n) = -(n+1)$ if $n$ is odd.

The base case is $n = 0$, where we have

$$f(0) = 0 = (-1)^0(2\cdot 0),$$

so $P(0)$ is true.

The inductive step is to prove that for all $n \in \mathbb{N}$, if $P(n)$ is true then $P(n+1)$ is true. Let $n \in \mathbb{N}$ be arbitrary, and assume $P(n)$ is true. There are two cases, depending on whether $n$ is even or odd.

Case: $n$ is even. Then, by the fact that $P(n)$ is true, we have $f(n) = n$. Note that

$$f(n + 1) = f(n) + (-1)^{n+1}(2(n+1)).$$

Since $n + 1$ is odd, $(-1)^{n+1} = -1$, so this implies

$$f(n + 1) = n - 2(n + 1) = -n - 2 = -(n + 1) + 1.$$ 

Hence, $P(n + 1)$ is true in this case.

Case: $n$ is odd. Then, by the fact that $P(n)$ is true, we have $f(n) = -(n+1)$. Then

$$f(n + 1) = f(n) + (-1)^{n+1}2(n+1),$$

and $n + 1$ is even, so $(-1)^{n+1} = 1$ and

$$f(n + 1) = -(n + 1) + 2(n + 1) = n + 1,$$

which shows that $P(n + 1)$ is also true in this case. Thus, the inductive step is proved.

2. We give an inductive definition of an addition formula (AF) as follows.
Base case: \( x, y, \) and \( z \) are AFs.

Inductive case: if \( F_1 \) and \( F_2 \) are AFs, then so are \( (F_1 + F_2) \) and \( (-F_1) \).

We inductively define the function \( v \) from AFs to \( \mathbb{N} \) as follows.

\[
v(x) = v(y) = v(z) = 1.
\]

If \( F_1 \) and \( F_2 \) are AFs, then

\[
v((F_1 + F_2)) = v(F_1) + v(F_2),
\]

and

\[
v((-F_1)) = v(F_1).
\]

We also inductively define the function \( d \) from AFs to \( \mathbb{N} \) as follows.

\[
d(x) = d(y) = d(z) = 0.
\]

If \( F_1 \) and \( F_2 \) are AFs, then

\[
d((F_1 + F_2)) = 1 + \max(d(F_1), d(F_2)),
\]

and

\[
d((-F_1)) = 1 + d(F_1).
\]

(a) (10 points) Give five examples of AFs, and for each AF \( F \), list the values of \( v(F) \) and \( d(F) \). Make sure no two of your example formulas have the same ordered pair of \((v(F), d(F))\).

**Solution.**

<table>
<thead>
<tr>
<th>( F )</th>
<th>( x )</th>
<th>((-x))</th>
<th>((x + y))</th>
<th>((x + (-y)))</th>
<th>((x + (y + z)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(F) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( d(F) )</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

(b) (15 points) Prove by structural induction that for all AFs \( F \), \( v(F) \leq 2^{d(F)} \).

**Solution.** The predicate \( P(F) \) is \( v(F) \leq 2^{d(F)} \). One base case is \( F = x \), for which we have \( v(x) = 1 \) and \( d(x) = 0 \), so \( v(x) \leq 2^{d(x)} \). The cases of \( F = y \) and \( F = z \) are the same.

The inductive step is to show that for all AFs \( F_1 \) and \( F_2 \), if \( P(F_1) \) and \( P(F_2) \) are true, then both \( P((F_1 + F_2)) \) and \( P((-F_1)) \) are true. To prove this, we let \( F_1 \) and \( F_2 \) be arbitrary AFs and assume that \( P(F_1) \) and \( P(F_2) \) are true, that is, that \( v(F_1) \leq 2^{d(F_1)} \) and \( v(F_2) \leq 2^{d(F_2)} \). We consider the cases of \( (F_1 + F_2) \) and \((-F_1)\) separately.
Case: for $F = (F_1 + F_2)$, we have $v(F) = v(F_1) + v(F_2)$, and therefore
$$v(F) \leq 2^{d(F_1)} + 2^{d(F_2)}.$$ 
Also, $d(F) = 1 + \max(d(F_1), d(F_2))$. Now
$$d(F_1) \leq \max(d(F_1), d(F_2)),$$
so
$$2^{d(F_1)} \leq 2^{\max(d(F_1), d(F_2))},$$
and, similarly,
$$2^{d(F_2)} \leq 2^{\max(d(F_1), d(F_2))},$$
which implies that
$$2^{d(F_1)} + 2^{d(F_2)} \leq 2 \cdot 2^{\max(d(F_1), d(F_2))} = 2^{d(F)},$$
and therefore $v(F) \leq 2^{d(F)}$.

Case: for $F = (-F_1)$, we have $v(F) = v(F_1)$ and therefore
$$v(F) \leq 2^{d(F_1)}.$$
Because $d(F) = 1 + d(F_1)$, we have $2^{d(F_1)} < 2^{d(F)}$, and therefore $v(F) \leq 2^{d(F)}$ in this case also, concluding the proof of the inductive step.

3. (a) (10 points) We define the functions $f$ and $g$ with domain and co-domain $\mathbb{N}$ by $f(n) = n^3$ and $g(n) = n^2 + 51$ for every $n \in \mathbb{N}$. Prove from the definitions in Section 7.1 of the course text that $g(n)$ is in $O(f(n))$ but $g(n)$ is not in $\Omega(f(n))$.

Solution. To prove that $g(n)$ is in $O(f(n))$, we must show that there exist $c > 0$ and $N$ such that for all $n \in \mathbb{N}$, if $n > N$ then
$$|g(n)| \leq c|f(n)|.$$ 
We may choose $c = 2$ and $N = 4$. For every $n \in \mathbb{N}$ such that $n > N$, we have $n^2 \leq n^3$ and $64 \leq n^3$, so
$$|g(n)| = n^2 + 51 \leq n^3 + n^3 = 2n^3 = c|f(n)|,$$
which shows that $g(n)$ is in $O(f(n))$.
To see that $g(n)$ is not in $\Omega(f(n))$, we must show that for every $c > 0$ and $N$, there exists $n \in \mathbb{N}$ such that $n > N$ and
$$|g(n)| < c|f(n)|.$$
Let $c > 0$ and $N$ be arbitrary. Choose any $n \in \mathbb{N}$ such that $n > N$, $n \geq 8$ and $n > 2/c$. (One plus the maximum of the three numbers would suffice.) For this $n$ we have $n^2 \geq 64$ and

$$c|f(n)| = cn^3 > c(2/c)n^2 = 2n^2 = n^2 + n^2 > n^2 + 51 = |g(n)|,$$

which shows that $g(n)$ is not in $\Omega(f(n))$.

(b) (15 points) For every $n \in \mathbb{N}$, we define

$$h(n) = \sum_{i=0}^{n} i^2 3^i.$$

Prove that $h(n)$ is in $\Theta(n^2 3^n)$.

Solution. To see that $h(n)$ is in $\Omega(n^2 3^n)$, note that the largest term in the sum defining $h(n)$ is $n^2 3^n$ and all the other terms are nonnegative, so for all $n \geq 0$ we have

$$h(n) \geq n^2 3^n,$$

witnessing $h(n)$ in $\Omega(n^2 3^n)$ with $N = 0$ and $c = 1$.

To see that $h(n)$ is in $O(n^2 3^n)$, note that for each $i = 0, 1, \ldots, n$,

$$i^2 3^i \leq n^2 3^i.$$

Then for each $n \in \mathbb{N}$ we have (by Lemma 6.1.2)

$$h(n) \leq \sum_{i=0}^{n} n^2 3^i = n^2 \sum_{i=0}^{n} 3^i.$$

Using a “standard sum” from section 6.4.1, we have

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2} \leq 3 \cdot 3^n.$$

Thus, for all $n \geq 0$,

$$h(n) \leq 3n^2 3^n,$$

witnessing that $h(n)$ is in $O(n^2 3^n)$ with $c = 3$ and $N = 0$. Because $h(n)$ is in $O(n^2 3^n)$ and also in $\Omega(n^2 3^n)$, we have that $h(n)$ is in $\Theta(n^2 3^n)$ (section 7.1).

4. (a) (10 points) Let $p$ and $q$ be arbitrary positive integers such that $p < q$. Let $n = \lceil(q/p)\rceil$. (See section 3.5.1 if you don’t know the definition of $\lceil x \rceil$..) Note that by fraction arithmetic

$$\frac{p}{q} - \frac{1}{n} = \frac{np - q}{qn}.$$
Prove that $0 \leq np - q < p$. (Hint: for all real numbers $x$, $x \leq \lfloor x \rfloor < x + 1$.)

**Solution.** Using the hint, we have

$$\frac{q}{p} \leq n < \frac{q}{p} + 1,$$

and multiplying through by $p$ we have

$$q \leq np < q + p.$$

Subtracting $q$, we get

$$0 \leq np - q < p,$$

which was to be proved.

(b) (10 points) If $p$ is a positive integer, define the predicate $R(p)$ to be true if and only if for all positive integers $q$, if $p < q$ then $p/q$ can be expressed as a finite sum of one or more distinct fractions of the form $1/m$ for some integers $m \geq 2$. (For example, $3/7 = 1/3 + 1/11 + 1/231$.) Clearly, $R(1)$ is true. Prove that $R(2)$ is true. (Hint: try out some values of $q > 2$.)

**Solution.** Let $q$ be an arbitrary positive integer and assume $q > 2$. We consider two cases, depending on whether $q$ is even or odd.

Case: if $q$ is even then $q = 2r$ for some positive integer $r \geq 2$ and

$$\frac{2}{q} = \frac{2}{2r} = \frac{1}{r},$$

so $2/q$ is a sum of one fraction of the required form.

Case: if $q$ is odd, then $q = 2r + 1$ for some positive integer $r \geq 1$. Then

$$\frac{2}{q} = \frac{2}{2r + 1} = \frac{1}{r + 1} + \frac{1}{(2r + 1)(r + 1)},$$

so $2/q$ is a sum of two distinct fractions of the required form. Thus, in either case, $2/q$ is a finite sum of one or more fractions of the required form, and we conclude that $R(2)$ is true.

(c) (5 points) Prove by strong induction that $R(p)$ is true for all positive integers $p$. (You may assume the results in parts (a) and (b).)

**Solution.** The base cases of $R(1)$ and $R(2)$ are established in part (b). The inductive step is to prove that for all positive integers $p - 1$, if $R(1) \land R(2) \land \ldots \land R(p - 1)$ is true, then $R(p)$ is true. Let $p - 1$ be an arbitrary positive integer, and assume that $R(s)$ is true for $s = 1, 2, \ldots, p - 1$. We need to prove that for every positive integer $q > p$, the fraction $p/q$ can be expressed in the required way. Let $q$ be an arbitrary positive integer and assume that $q > p$. 

6
Let \( n = \lceil (q/p) \rceil \) and \( n \geq 2 \). Then

\[
\frac{q}{p} = \frac{1}{n} + \frac{np - q}{qn},
\]

and by part (a), \( 0 \leq (np - q) < p \). We consider cases based on whether \((np - q) = 0\) or not.

Case: if \((np - q) = 0\), then \( q/p = 1/n \), of the required form.

Case: if \( 1 \leq (np - q) \), then by the induction hypothesis, \( R(np - q) \) is true. Because \((np - q) < p < q < qn\), this means that the fraction \((np - q)/qn\) can be expressed as a finite sum of one or more distinct fractions of the form \( 1/m \) for some integers \( m \geq 2 \).

Because \((np - q) < q\), \((np - q)/qn < 1/n\), so all the fractions \( 1/m \) in the sum for \((np - q)/qn\) are distinct from \( 1/n\), and we may use \( 1/n \) and the fractions in the sum for \((np - q)/qn\) to express \( p/q \) in the required form. Since in either case, \( p/q \) is expressed in the required form, \( R(p) \) is true and the inductive step is proved.