1. (a) (10 points) Prove that for all positive integers $n$

\[ \sum_{i=1}^{n} (2i + 3) = n^2 + 4n. \]

For parts (b) and (c), we define $f : \mathbb{N} \to \mathbb{Z}$ by

\[ f(n) = \sum_{i=0}^{n} (-1)^i (2i). \]

(b) (5 points) Make a table of the values of $f(n)$ for $n = 0, 1, 2, 3, 4$.

(c) (10 points) Prove that for all natural numbers $n$, if $n$ is even then $f(n) = n$.

(Hint: you might want to prove something stronger than this.)

2. We give an inductive definition of an addition formula (AF) as follows.

Base case: $x$, $y$, and $z$ are AFs.

Inductive case: if $F_1$ and $F_2$ are AFs, then so are $(F_1 + F_2)$ and $(-F_1)$.

We inductively define the function $v$ from AFs to $\mathbb{N}$ as follows.

\[ v(x) = v(y) = v(z) = 1. \]
If $F_1$ and $F_2$ are AFs, then
\[ v((F_1 + F_2)) = v(F_1) + v(F_2), \]
and
\[ v((-F_1)) = v(F_1). \]

We also inductively define the function $d$ from AFs to $\mathbb{N}$ as follows.
\[ d(x) = d(y) = d(z) = 0. \]
If $F_1$ and $F_2$ are AFs, then
\[ d((F_1 + F_2)) = 1 + \max(d(F_1), d(F_2)), \]
and
\[ d((-F_1)) = 1 + d(F_1). \]

(a) (10 points) Give five examples of AFs, and for each AF $F$, list the values of $v(F)$ and $d(F)$. Make sure no two of your example formulas have the same ordered pair of $(v(F), d(F))$.

(b) (15 points) Prove by structural induction that for all AFs $F$, $v(F) \leq 2^d(F)$.

3. (a) (10 points) We define the functions $f$ and $g$ with domain and co-domain $\mathbb{N}$ by $f(n) = n^3$ and $g(n) = n^2 + 51$ for every $n \in \mathbb{N}$. Prove from the definitions in Section 7.1 of the course text that $g(n)$ is in $O(f(n))$ but $g(n)$ is not in $\Omega(f(n))$.

(b) (15 points) For every $n \in \mathbb{N}$, we define
\[ h(n) = \sum_{i=0}^{n} i^2 3^i. \]
Prove that $h(n)$ is in $\Theta(n^2 3^n)$.

4. (a) (10 points) Let $p$ and $q$ be arbitrary positive integers such that $p < q$. Let $n = \lceil (q/p) \rceil$. (See section 3.5.1 if you don’t know the definition of $\lceil x \rceil$.) Note that by fraction arithmetic
\[ \frac{p}{q} - \frac{1}{n} = \frac{np - q}{qn}. \]
Prove that $0 \leq np - q < p$. (Hint: for all real numbers $x$, $x \leq \lceil x \rceil < x + 1$.)
(b) (10 points) If \( p \) is a positive integer, define the predicate \( R(p) \) to be true if and only if for all positive integers \( q \), if \( p < q \) then \( p/q \) can be expressed as a finite sum of one or more distinct fractions of the form \( 1/m \) for some integers \( m \geq 2 \). (For example, \( 3/7 = 1/3 + 1/11 + 1/231 \).) Clearly, \( R(1) \) is true. Prove that \( R(2) \) is true. (Hint: try out some values of \( q > 2 \).)

(c) (5 points) Prove by strong induction that \( R(p) \) is true for all positive integers \( p \). (You may assume the results in parts (a) and (b).)