Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. Partial credit will be given if and only if the grader can easily understand enough of your answer to award it. Include your name, your netid, the homework number and the problem number on EACH PART.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 1 point on this homework.

Graphs and trees in these problems are assumed to have at least one vertex. The degree sequence of a simple undirected graph is the sequence of degrees of its vertices, sorted into non-increasing order. For example, if $G = (V, E)$ is the simple undirected graph with vertices $V = \{1, 2, 3, 4, 5, 6\}$ and edges $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{4, 5\}\}$ then the degree sequence of $G$ is $(3, 2, 2, 2, 1, 0)$.

To construct proofs requested below, you may use definitions and results about graphs and trees given in Chapter 10 of “Notes on Discrete Mathematics” and things you prove yourself, but not facts about graphs and trees from elsewhere. Please cite by number the Lemmas and Theorems you use from Chapter 10. General hints: What lemma relates the degrees of vertices to the number of edges in a graph? What useful theorems about trees are in Chapter 10? What is the definition of a spanning tree?

1. Possible degree sequences.

(a) (10 points) For each of the following sequences, determine whether it is the degree sequence of some simple undirected graph with 6 vertices. If it is, draw one such graph; if it isn’t, prove that it is not.

i. (2, 2, 2, 1, 1, 1)
ii. (2, 2, 1, 1, 1, 1)
iii. (5, 3, 3, 2, 2, 1)
iv. (3, 3, 3, 2, 2)
v. (5, 4, 3, 3, 1, 0)
(b) (12 points) For each of the following sequences, determine (A) whether or not it is the degree sequence of some tree, AND (B) whether or not it is the degree sequence of some non-tree. If there is a tree, draw one such; if not, prove there is no such tree. And if there is a non-tree, draw one such; if not, prove there is no such non-tree. There may be one or the other, or both, or neither.

i. (3,3,2,2,1,1)

ii. (3,3,2,2,1,1,1)

iii. (4,3,2,1,1,1,1,1)

c) (3 points) Prove that if \( T \) is a tree that contains a simple path of length at least 4, then there is a graph \( G \) that is not a tree but has the same degree sequence as \( T \).

2. Given a graph \( G = (V, E) \), we define another graph \( G \square G \) that has vertices \( V' = V \times V \) and edges \( E' \) where for all \((u_1, u_2) \in V'\) and \((v_1, v_2) \in V'\), there is an edge in \( E' \) from \((u_1, u_2)\) to \((v_1, v_2)\) if and only if either \( u_1v_1 \in E \) and \( u_2 = v_2 \) or \( u_1 = v_1 \) and \( u_2v_2 \in E \).

(a) (15 points) Let \( G \) be the graph with vertices \( V = \{a, b, c\} \) and edges \( E = \{ab, bc, ca\} \). Draw diagrams of \( G \) and \( G \square G \), and list the vertices in a Hamiltonian cycle in \( G \square G \).

(b) (10 points) Prove that for any graph \( G \), if there is a Hamiltonian cycle in \( G \), then there is a Hamiltonian cycle in \( G \square G \).

3. (a) (10 points) Use the Handshaking Lemma (Lemma 10.10.3) and Theorem 10.10.8 to prove that if \( G = (V, E) \) is a tree with at least two vertices, then \( G \) has at least two vertices of degree equal to 1. (Hint: a graph with too many edges cannot be a tree.)

(b) (15 points) Let \( P(n) \) be the predicate that every tree with \( n \) vertices can be constructed by starting with a single vertex and no edges, and then repeating \((n - 1)\) times the operation of adding a new vertex and an edge joining that new vertex to one of the existing vertices. Prove by mathematical induction that for all positive integers \( n \), \( P(n) \) is true. Explicitly label the base case(s) and the statement of the inductive step. (Hint: you may assume part (a) is true, and you may find Lemma 10.10.5 useful.)

4. Consider the following algorithm that takes as input a graph \( G = (V, E) \) and constructs a graph \( H = (V, E') \), where \( E' \subseteq E \).

Set \( E' = \emptyset \).

For each \( uv \in E \)

If the graph \((V, E' \cup \{uv\})\) is acyclic, then set \( E' = E' \cup \{uv\} \).
Output $H = (V, E')$.

(a) (10 points) Run the algorithm on the graph $G$ with vertices $\{a, b, c, d, e\}$ and edges $\{ab, bc, ca, cd, da, ce, de\}$, showing the successive values of $E'$.

(b) (10 points) Prove that if the input graph $G = (V, E)$ is connected, then the output $H = (V, E')$ is a spanning tree of $G$. (See Sect. 10.10.4 on spanning trees.)

(c) (5 points) Describe what the algorithm of this problem computes if the input graph $G$ is not connected. (No proof is necessary.)