Note that the “Late” period for this homework is until the end of the lecture on Tuesday, Dec. 3.

Please turn in your homework in FOUR SEPARATE PARTS, one for each problem. It may be handwritten (pen or pencil) or computer typeset. Include your name, your netid, the homework number and the problem number on EACH PART. Partial credit will be given if and only if the grader can easily understand enough of your answer to award it.

You MAY collaborate on the homework. However, with the first part, you must write the names of all the persons (course staff and others) you have talked with about the assignment, and identifying information for all the sources (online or not), other than the course text, that you have consulted in connection with the assignment. Write “Collaborators: None” or “Sources: None” if you have none to report. Failure to do this will cost 1 point on this homework.

For this assignment, while you may consult outside resources, any facts that you use in your proofs must be cited from the course text, Notes on Discrete Math, rather than outside resources.

1. (25 points) Let $A$ be the set of base 6 representations of the first two hundred and sixteen nonnegative integers. Thus

$$A = \{0, 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 30, 31, \ldots \}.$$ 

Then $|A| = 216$, and only the representation 0 begins with the digit 0. We consider occurrences of digits in elements of $A$. For example, in the representation 115, the digit 1 occurs twice and the digit 5 occurs once. The length of a representation is the number of occurrences of any digit in it; thus, the length of 115 is 3. Answer the following questions (in base ten) about $A$, giving reasons for your answers (including, as appropriate, the sum rule, the difference rule, the product rule, and so on, from Chapter 11.)

(a) How many elements of $A$ have length 1, length 2, length 3, and length 4? What is the sum of the lengths of the elements of $A$?

(b) How many elements of $A$ have no occurrences of 3? (Hint: you may want to consider them partitioned by length.)

(c) How many elements of $A$ have at least one occurrence of 1 or 4?

(d) How many occurrences of 4 are there in all the elements of $A$?

(e) How many occurrences of 0 are there in all the elements of $A$?

2. We develop a method to determine how many different arrangements there are of the letters in the word “stateliness”.

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(a) (10 points) First imagine that every occurrence of a letter in the word is subscripted with the number of occurrences of that letter that precede the current occurrence. That is, the first occurrence of $s$ is $s_0$, the second is $s_1$, and the third is $s_2$. Write out the result of applying this subscripting rule to “stateliness” and determine how many different arrangements there are of these subscripted letters, treating the same letter with different subscripts as different.

(b) (10 points) Show that for any one arrangement of the subscripted letters in part (a), there are exactly 23 additional arrangements that produce the same sequence of letters when the subscripts are erased. Use this to determine the answer to the question of how many different arrangements there are of the letters in the word “stateliness”.

(c) (5 points) Give the general principle for how to compute the number of different arrangements of letters in a word of length $n$ letters, where there are $m$ multiply-occurring letters, which occur $k_1, k_2, \ldots, k_m$ times, respectively.

3. You are in the bagel store to get a dozen (12) bagels. Suppose the store has just 4 kinds of bagels: onion, plain, raisin and sesame. Suppose that there are at least 12 of each kind of bagel available for purchase. Answer the following questions, giving reasons for your answers.

(a) (10 points) How many different collections of 12 bagels can you get? A collection is specified by the number of each kind of bagel you get, for example: 5 onion, 1 plain, 2 raisin and 4 sesame. (Hint: find a correspondence between collections of 12 bagels and strings of 15 bits containing exactly 3 occurrences of 1, and give the string of bits corresponding to the listed example collection.)

(b) (10 points) How many different collections of 12 bagels can you get, assuming that you must get at least 2 plain bagels and at least 3 onion bagels?

(c) (5 points) What is the general formula for the number of collections of $n$ bagels if there are $k$ kinds of bagels and there are at least $n$ bagels of each kind? Explain your answer.

4. For any positive integer $n$, we define $S_n = \{1, 2, 3, \ldots, n\}$. This problem concerns the following claim.

**Claim:** For any positive integer $n$, the number of subsets of $S_n$ with even cardinality is equal to the number of subsets of $S_n$ with odd cardinality.

(a) (5 points) For $n = 6$, find the number of subsets of $S_6$ with each cardinality $k = 0, 1, 2, 3, 4, 5, 6$, and verify that the above Claim is true in this case.

(b) (5 points) Use the finite version of the Binomial Theorem (equation (11.2.3)) to prove the above Claim for an arbitrary positive integer $n$. 

(c) (5 points) Let $n$ be an arbitrary positive integer. Recall that $\mathcal{P}(S_n)$ is the set of all subsets of $S_n$. We define a function $f : \mathcal{P}(S_n) \rightarrow \mathcal{P}(S_n)$ as follows. For any subset $T$ of $S_n$, $f(T) = T \setminus \{1\}$ if $1 \in T$ and $f(T) = T \cup \{1\}$ if $1 \notin T$. For the case of $n = 3$, list the value of $f(T)$ for each of the 8 subsets $T$ of $S_3$.

(d) (10 points) Prove that for any positive integer $n$, the function $f$ defined in part (c) is a bijection, and use that result to give another proof of the above Claim for any positive integer $n$. 