YOUR NAME PLEASE:  
***** SOLUTIONS *****

Computer Science 202  
Practice Midterm  
October 2019

Show ALL work you want graded ON THE TEST ITSELF. 
For problems that do not ask you to justify the answer, an answer alone is sufficient. However, if the answer is wrong and no derivation or supporting reasoning is given, there will be no partial credit.

GOOD LUCK!
1. (10 points) Give a succinct definition of each of the following terms. Assume $A$, $B$, and $C$ are sets, $f$ is a function with domain $A$ and co-domain $B$, and $g$ is a function with domain $B$ and co-domain $C$.

(a) The union of $A$ and $B$, denoted $A \cup B$.

Solution.

$A \cup B = \{ x \mid x \in A \vee x \in B \}$.

(Or: $A \cup B$ is the set of all elements $x$ such that $x$ is in $A$ or $x$ is in $B$.)

(b) The function $f$ is bijective.

Solution. For every $b \in B$ there exists $a \in A$ such that $f(a) = b$ and for every $a_1, a_2 \in A$, if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.

(c) The Cartesian product of $A$ and $B$, denoted $A \times B$.

Solution.

$A \times B = \{ (a, b) \mid a \in A \land b \in B \}$.

(Or: $A \times B$ is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$.)

(d) The composition of $g$ and $f$, denoted $g \circ f$.

Solution. $g \circ f$ is the function with domain $A$ and co-domain $C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$.

(e) The power set of $A$, denoted $\mathcal{P}(A)$.

Solution.

$\mathcal{P}(A) = \{ S \mid S \subseteq A \}$.

(Or, $\mathcal{P}(A)$ is the set of all subsets of $A$.)
2. (10 points) Determine whether the following statements are true for ALL sets of positive integers \( A, B, \) and \( C \). Your answer should be a proof or a counterexample, NOT a Venn diagram.

(a) \((A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus (A \cap B \cap C)\).

Solution. This is not true of all sets of positive integers \( A, B, \) and \( C \). Consider \( A = \{1, 2, 3\}, B = \{2, 3, 5\} \) and \( C = \{3, 4, 5\} \).

Then \((A \setminus B) = \{1\}\) and \((C \setminus B) = \{4\}\), so their union (the lefthand side) is \(\{1, 4\}\). However, \((A \cup C) = \{1, 2, 3, 4, 5\}\), and \(A \cap B \cap C = \{3\}\), so their set difference (the righthand side) is \(\{1, 2, 4, 5\}\), which is not equal to the lefthand side. Thus for these values
\[(A \setminus B) \cup (C \setminus B) \neq (A \cup C) \setminus (A \cap B \cap C).\]

(b) If \( A \cap C \subseteq B \cap C \) and \( A \cup C \subseteq B \cup C \) then \( A \subseteq B \).

Solution. This is true for all sets of positive integers \( A, B \) and \( C \). Let \( A, B, C \) be arbitrary sets of positive integers, and assume that \( A \cap C \subseteq B \cap C \) and \( A \cup C \subseteq B \cup C \).

To see that \( A \subseteq B \), let \( a \) be an arbitrary element of \( A \). We consider two cases depending on whether \( a \in C \).

Case 1: If \( a \in C \), then \( a \in (A \cap C) \), which by the assumption that \((A \cap C) \subseteq (B \cap C)\) implies that \( a \in (B \cap C) \), and therefore \( a \in B \).

Case 2: If \( a \notin C \), then because \( a \in A \), we have \( a \in (A \cup C) \). By the assumption that \((A \cup C) \subseteq (B \cup C)\), we have \( a \in (B \cup C) \). But in this case, \( a \notin C \), so it must be that \( a \in B \).

In either case, \( a \in B \), and since \( a \in A \) was arbitrary, we conclude that \( A \subseteq B \).
3. (16 points) For each of the following functions from the positive integers to the positive integers, state whether it is injective and whether it is surjective, and justify your answers.

(a) \( f(n) \) is the largest odd integer that divides \( n \).

Solution. The function \( f \) is not injective, because \( f(2) = 1 = f(4) \). It is not surjective, because \( f(n) \) is always odd, so for all positive integers \( n \), \( f(n) \neq 2 \).

(b) \( g(n) = n^2 \).

Solution. The function \( g \) is not surjective, because 2 is not the square of any integer, so \( g(n) \neq 2 \) for all positive integers \( n \). It is injective, because if \( g(n) = g(m) \) for some positive integers \( n \) and \( m \), we have \( n^2 = m^2 \), so

\[
(n - m) \cdot (n + m) = 0.
\]

Since \( n + m \neq 0 \), this implies that \( n - m = 0 \), that is, \( n = m \).

(c) \( h(n) \) is the sum of the digits in the decimal representation of \( n \).

Solution. The function \( h \) is not injective because \( h(21) = 3 = h(12) \). It is surjective, because for any positive integer \( n \), for the integer \( m \) represented in decimal by \( n \) 1’s in a row, we have \( h(m) = n \). For example, \( h(1) = 1 \), \( h(11) = 2 \), \( h(111) = 3 \), and so on.

(d) \( k(n) = \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \).

Solution. [Hint: calculate \( k(n) \) for \( n \) from 1 to 6.] The function \( k \) is not injective, because

\[
k(4) = \left\lfloor \frac{4}{6} \right\rfloor + \left\lfloor \frac{4}{3} \right\rfloor + \left\lfloor \frac{4}{2} \right\rfloor = 1 + 1 + 2 = 4,
\]

and

\[
k(5) = \left\lfloor \frac{5}{6} \right\rfloor + \left\lfloor \frac{5}{3} \right\rfloor + \left\lfloor \frac{5}{2} \right\rfloor = 1 + 1 + 2 = 4.
\]

It is also not surjective, because \( k \) is nondecreasing and

\[
k(6) = \left\lfloor \frac{6}{6} \right\rfloor + \left\lfloor \frac{6}{3} \right\rfloor + \left\lfloor \frac{6}{2} \right\rfloor = 1 + 2 + 3 = 6,
\]

so there is no positive integer \( n \) such that \( k(n) = 5 \).
4. For any positive integer \( n \), we define

\[
f(n) = \sum_{i=1}^{n} i \cdot i!.
\]

(a) (3 points) List the values of \( f(n) \) for \( n = 1, 2, 3 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>23</td>
</tr>
</tbody>
</table>

(b) (12 points) Prove the following statement by mathematical induction, identifying the predicate \( P(n) \), the base case(s) and the statement of the inductive step.

For every positive integer \( n \), \( f(n) = (n + 1)! - 1 \).

Solution. The predicate \( P(n) \) is the statement

\[
f(n) = (n + 1)! - 1.
\]

The base case is \( P(1) \), which is true because

\[
f(1) = 1 = 2 - 1 = (1 + 1)! - 1.
\]

The statement of the inductive step is that for all positive integers \( n \), \( P(n) \) implies \( P(n + 1) \), or

\[
(\forall n \geq 1)(P(n) \rightarrow P(n + 1)).
\]

To prove this, let \( n \) be an arbitrary positive integer, and assume that \( P(n) \) is true. That is, \( f(n) = (n + 1)! - 1 \). By the definition of \( f \), we have

\[
f(n + 1) = \sum_{i=1}^{n+1} i \cdot i!,
\]

which, if we separate the last term, is

\[
f(n + 1) = (n + 1) \cdot (n + 1)! + f(n).
\]

Using the assumption that \( P(n) \) is true, we can replace \( f(n) \) by \( (n + 1)! - 1 \):

\[
f(n + 1) = (n + 1) \cdot (n + 1)! + (n + 1)! - 1,
\]

which by distributivity implies

\[
f(n + 1) = (n + 2) \cdot (n + 1)! - 1 = (n + 2)! - 1,
\]

that is, \( P(n + 1) \) is true. This concludes the proof of the inductive step.
5. We define \( g(1) = 1 \) and \( g(2) = 3 \) and for each positive integer \( n \geq 3 \), we define

\[
g(n) = g(n-1) + g(n-2).
\]

(a) (5 points) List the values of \( g(n) \) for \( n = 1, 2, 3, 4, 5 \).

Solution. 

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(n) )</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

(b) (10 points) Prove by mathematical induction that for all positive integers \( n \), \( g(n) < 2^n \).

Solution. We prove the claim by strong induction. The predicate \( P(n) \) is

\[
g(n) < 2^n.
\]

The base cases are \( P(1) \) and \( P(2) \). \( P(1) \) is true because \( g(1) = 1 < 2 = 2^1 \), and \( P(2) \) is true because \( g(2) = 3 < 4 = 2^2 \).

The statement of the inductive step [strong induction] is that for all positive integers \( n \geq 2 \), if \( P(1), P(2), \ldots, P(n) \) are all true, then \( P(n+1) \) is also true.

Let \( n \) be an arbitrary positive integer and assume that \( n \geq 2 \) and that \( P(1), P(2), \ldots, P(n) \) are all true. Because \( n \geq 2 \), by the definition of \( g \), we have

\[
g(n+1) = g(n) + g(n-1).
\]

Because \( 1 \leq n - 1 \leq n \), our assumptions imply that \( P(n-1) \) and \( P(n) \) are true. Thus,

\[
g(n-1) < 2^{n-1},
\]

and

\[
g(n) < 2^n.
\]

Summing these two inequalities, we have

\[
g(n+1) = g(n) + g(n-1) < 2^n + 2^{n-1} < 2^{n+1}.
\]

Thus, \( P(n+1) \) is true, and the inductive step is concluded.
6. We consider a domain that consists of all the animals in a certain zoo. We define the following predicates.

\[ E(x) \text{ means } x \text{ is an elephant} \]
\[ M(x) \text{ means } x \text{ is a monkey} \]
\[ G(x) \text{ means } x \text{ is a giraffe} \]
\[ K(x) \text{ means } x \text{ is kindly} \]
\[ H(x) \text{ means } x \text{ is honest} \]
\[ I(x) \text{ means } x \text{ is insincere} \]
\[ D(x) \text{ means } x \text{ is deep} \]

(a) With these predicates, write the following statements as logical formulas.

i. (2 points) Every giraffe is insincere.
   \[ Solution. \ (\forall x)(G(x) \rightarrow I(x)) \]

ii. (2 points) No kindly animal is insincere.
   \[ Solution. \ \neg(\exists x)(K(x) \land I(x)) \]

iii. (2 points) All monkeys are honest, but there is at least one that is not kindly.
   \[ Solution. \ (\forall x)(M(x) \rightarrow H(x)) \land (\exists x)(M(x) \land \neg K(x)) \]
iv. (2 points) Every insincere animal is not honest.

Solution. \((\forall x)(I(x) \to \neg H(x))\)

v. (2 points) The elephants are kindly but they're deep.

Solution. \((\forall x)(E(x) \to K(x) \land D(x))\)

vi. (2 points) Every kindly animal is honest.

Solution. \((\forall x)(K(x) \to H(x))\)

(b) (4 points) Does the last statement (vi) in part (a) follow logically from the other statements (i-v)? Please justify your answer.

Solution. Statement (vi) does not follow logically from statements (i)-(v). Consider the following Zoo, in which (i)-(v) are true but (vi) is false. The Zoo has exactly two animals, a monkey \(m_1\) and an elephant \(e_1\), which have the following properties.

<table>
<thead>
<tr>
<th>animal</th>
<th>E</th>
<th>M</th>
<th>G</th>
<th>K</th>
<th>H</th>
<th>I</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(e_1)</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Then statement (i) is vacuously true (because there are no giraffes). Statement (ii) is true: there are no kindly, insincere animals. Statement (iii) is true: all the monkeys (just \(m_1\)) are honest, and \(m_1\) is a monkey and not kindly. Statement (iv) is true vacuously (because there are no insincere animals). Statement (v) is true: all the elephants (just \(e_1\)) are both kindly and deep. However, statement (vi) is false, because \(e_1\) is kindly but not honest.
7. (10 points) Suppose $f_1$, $f_2$ and $g$ are functions from the natural numbers to the natural numbers. Prove that if $f_1(n)$ is in $O(g(n))$ and $f_2(n)$ is in $O(g(n))$, then the function $h(n)$ is in $O(g(n))$, where $h(n)$ is defined for all natural numbers $n$ by 

$$h(n) = f_1(n) + f_2(n).$$

Solution. Let $f_1$, $f_2$ and $g$ be arbitrary functions from the natural numbers to the natural numbers. Because the co-domain in each case is the natural numbers, we have $|f_1(n)| = f_1(n)$, $|f_2(n)| = f_2(n)$, $|g(n)| = g(n)$, and $|h(n)| = h(n)$ for all natural numbers $n$. Assume that $f_1(n)$ is in $O(g(n))$ and that $f_2(n)$ is in $O(g(n))$. By the definition of “big-Oh”, this implies that there exist constants $c_1 > 0$ and $N_1$, as well as constants $c_2 > 0$ and $N_2$ such that the following are true.

For all $n > N_1$, $f_1(n) \leq c_1 \cdot g(n)$.

For all $n > N_2$, $f_2(n) \leq c_2 \cdot g(n)$.

Thus, if we let $c = (c_1 + c_2)$ and $N = \max(N_1, N_2)$, then $c > 0$ and for all $n > N$, it follows that $n > N_1$ and $n > N_2$, and therefore

$$h(n) = f_1(n) + f_2(n) \leq c_1 \cdot g(n) + c_2 \cdot g(n) = (c_1 + c_2) \cdot g(n) = c \cdot g(n).$$

These constants $c > 0$ and $N$ thus witness the fact that $h(n)$ is in $O(g(n))$. 
