COMPARISON-BASED ALGORITHMS represented as DECISION TREES

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$a_1 : a_2$ compare 2 numbers

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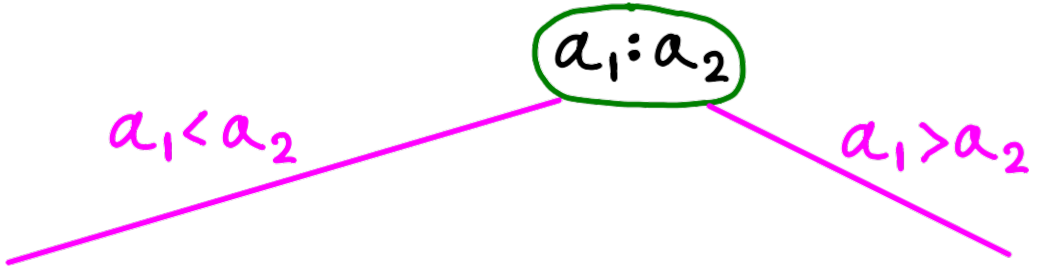

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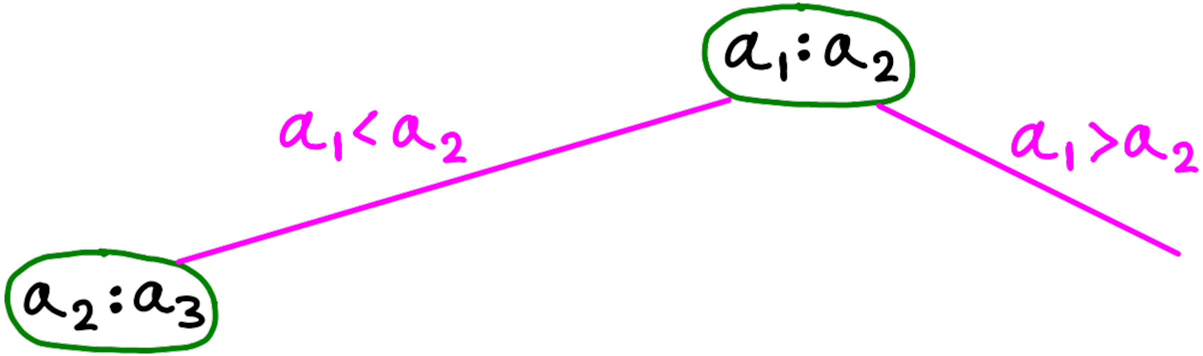

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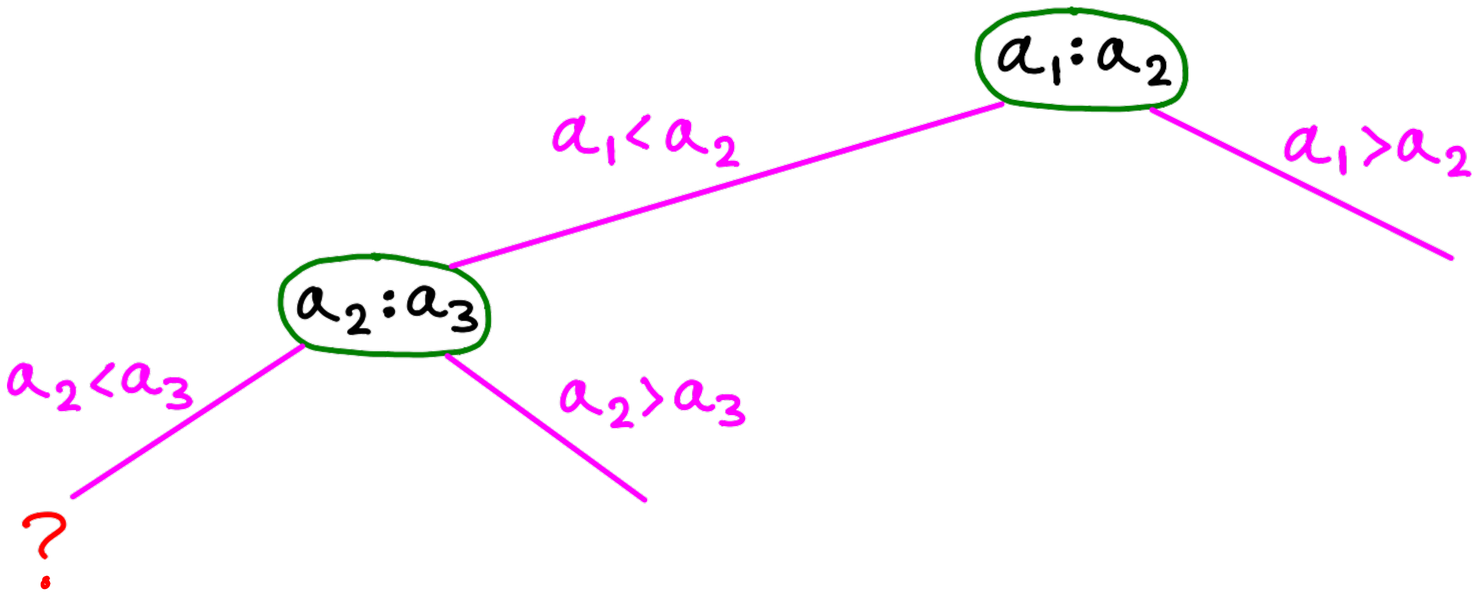

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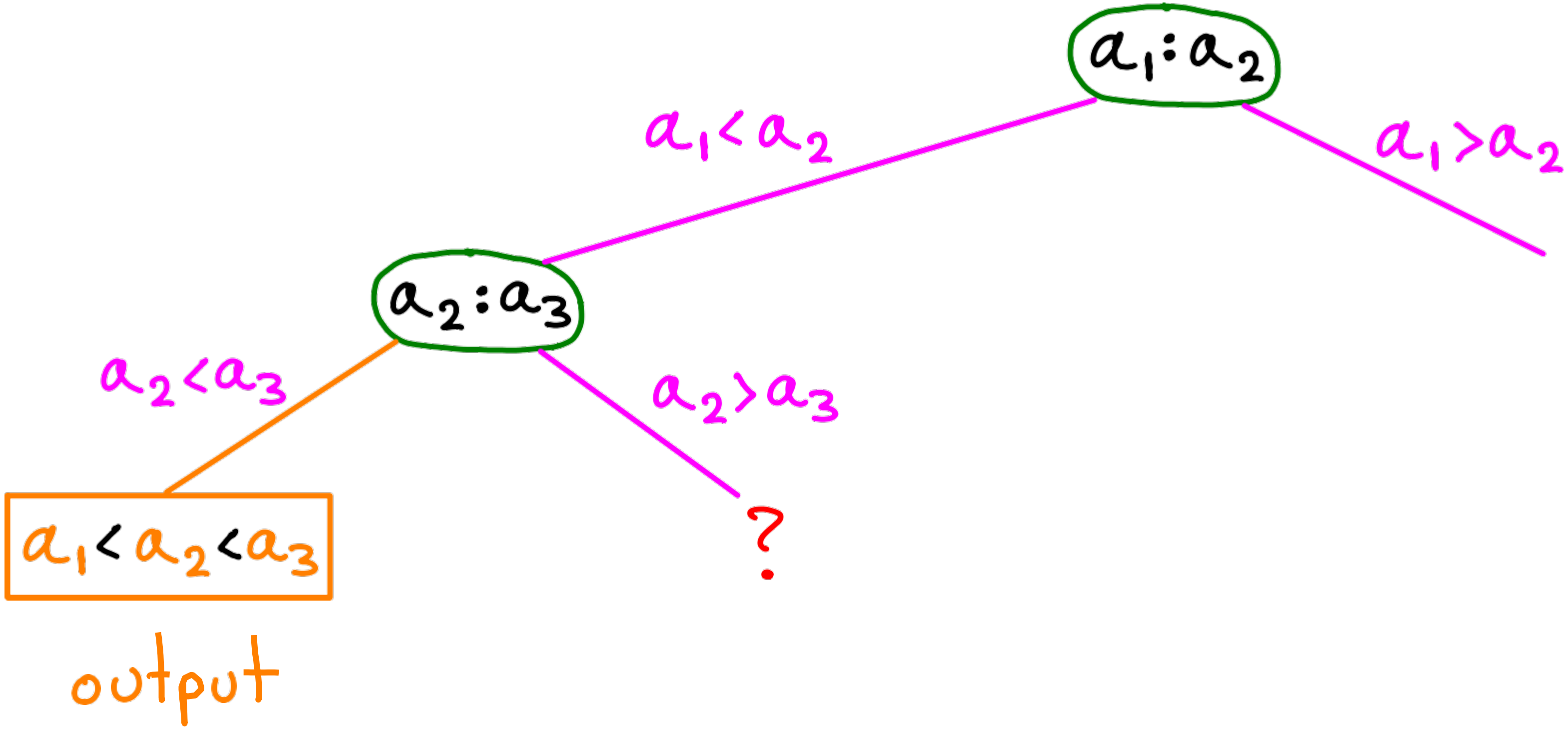

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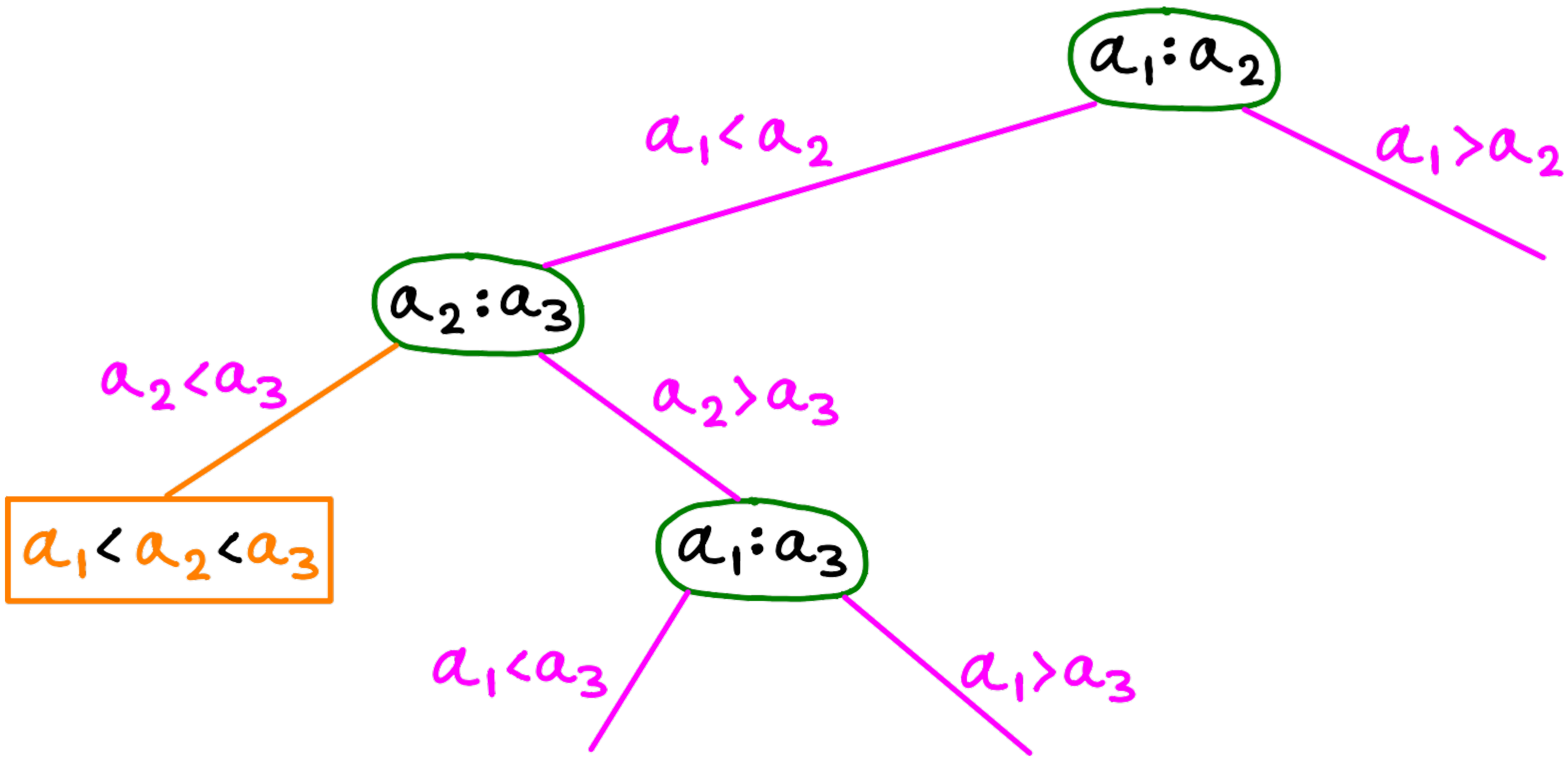

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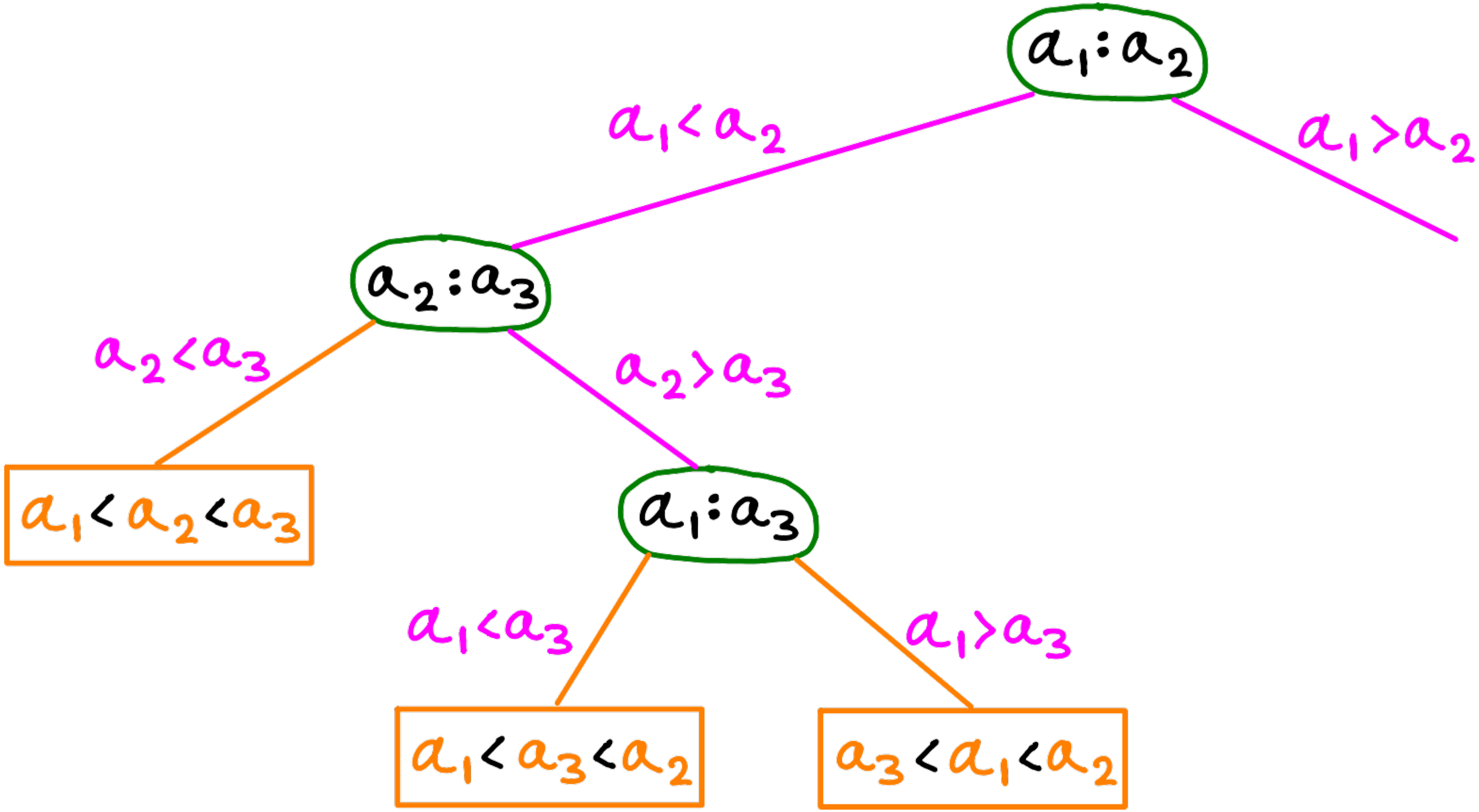

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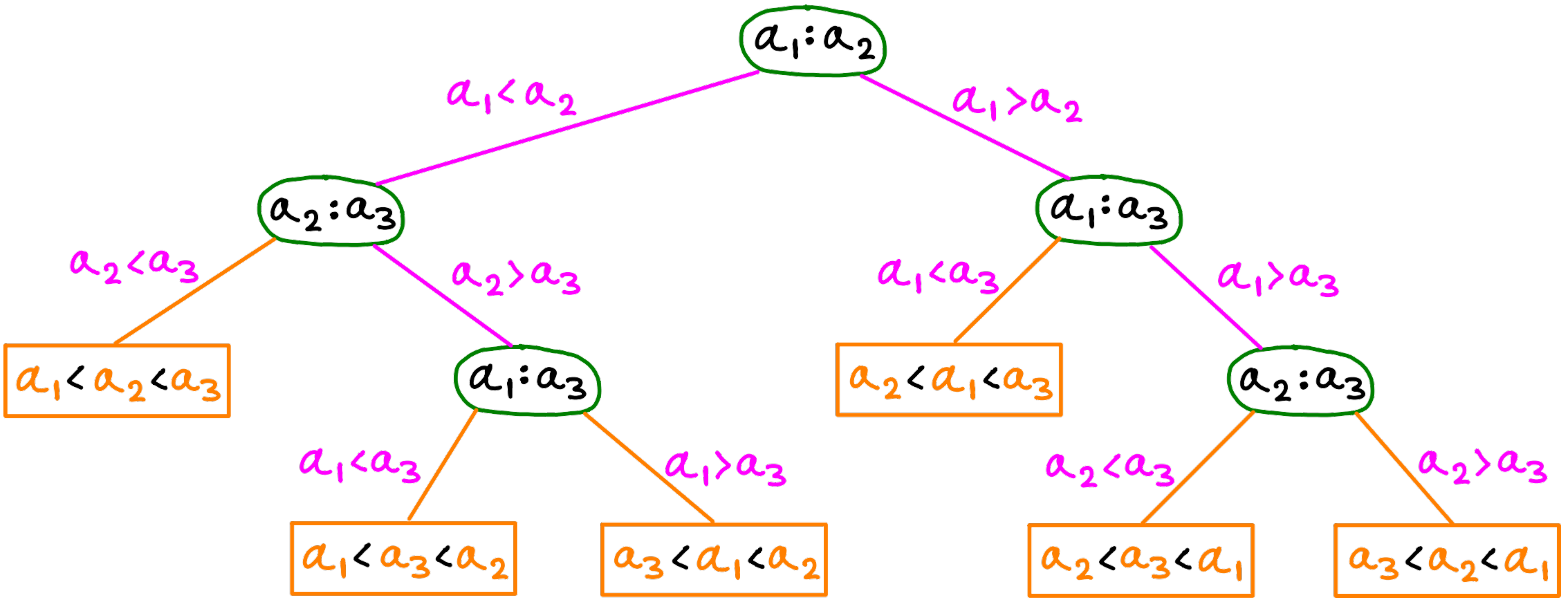
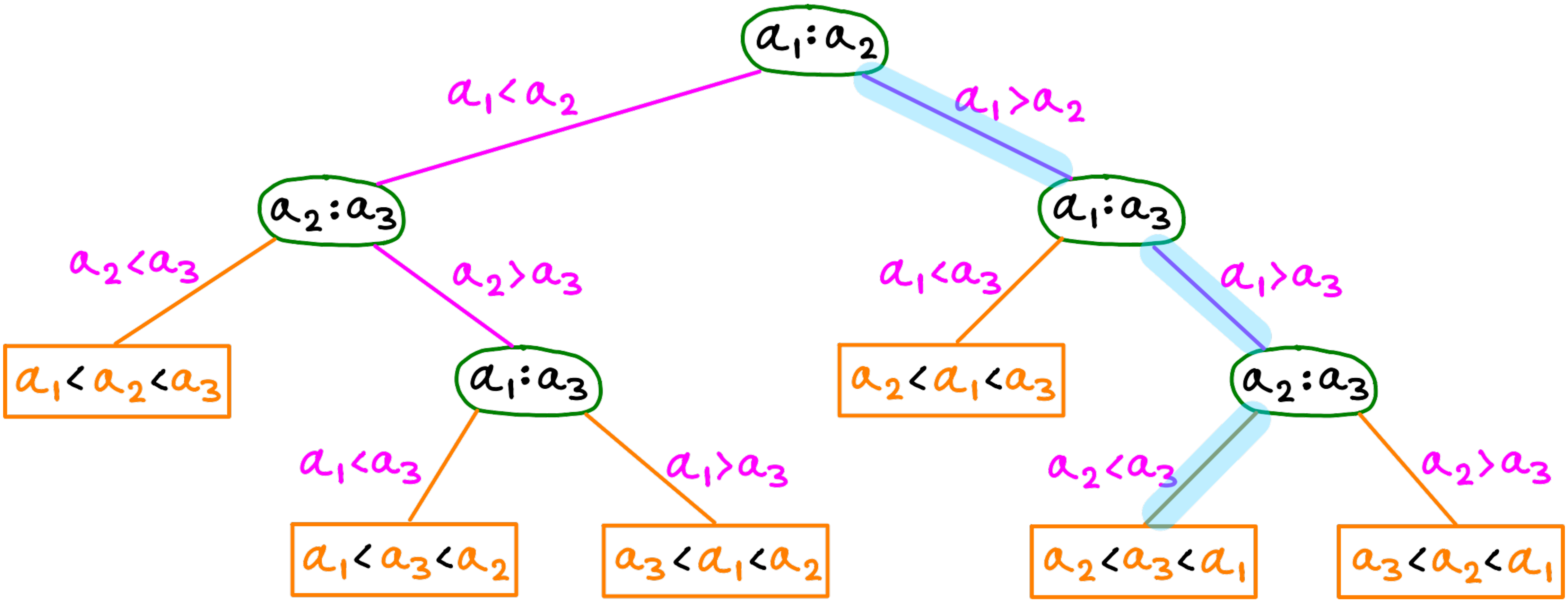


COMPARISON-BASED ALGORITHMS represented as DECISION TREES

example: an algorithm to sort 3 numbers: 9, 4, 6

$a_1 \quad a_2 \quad a_3$



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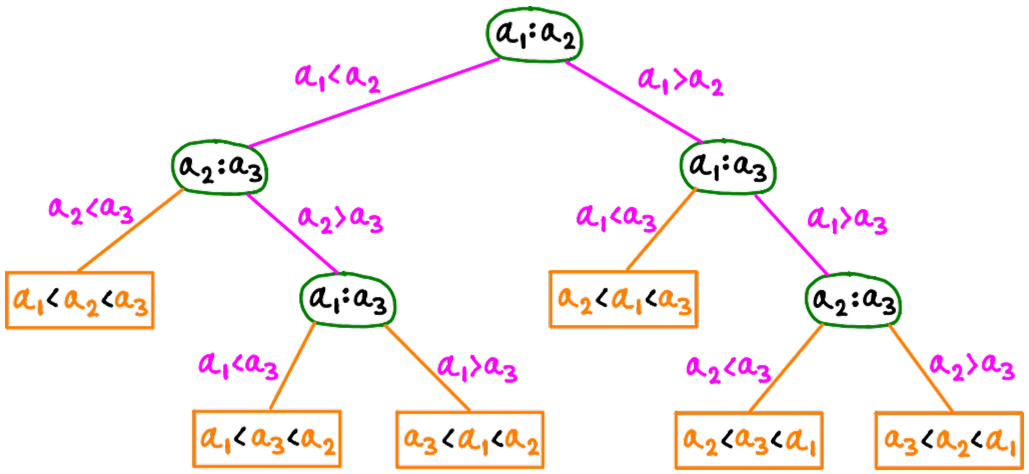

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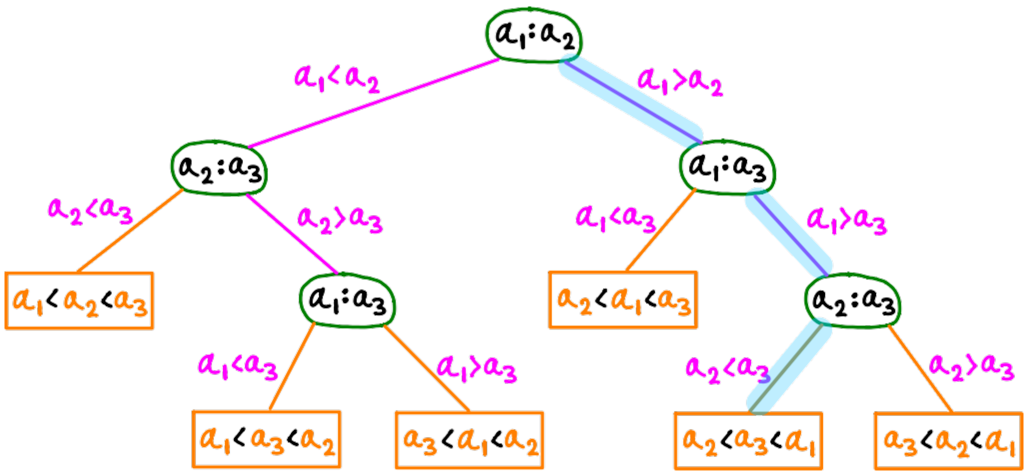

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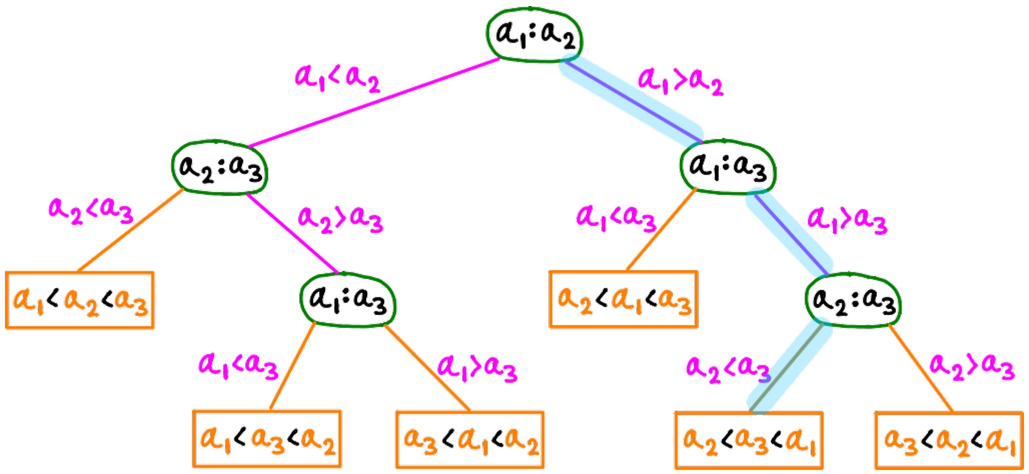

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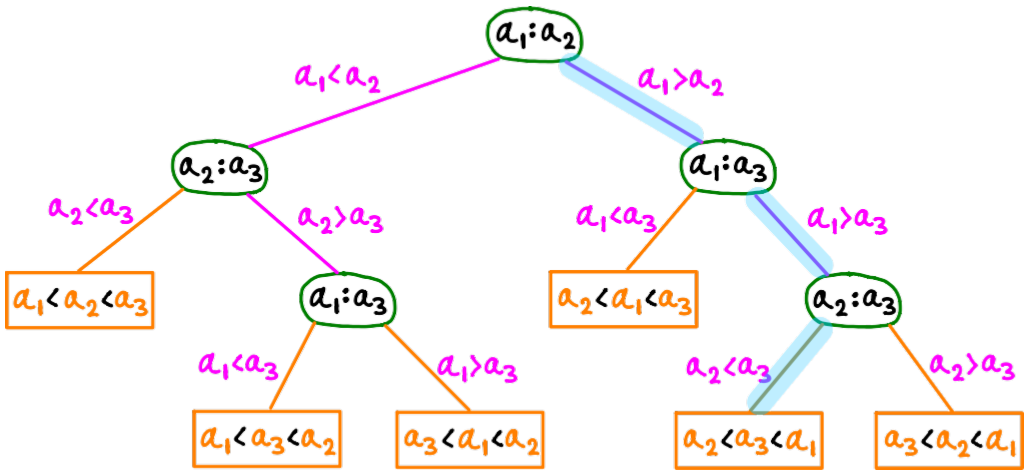

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$\boxed{\text{Longest path} \rightarrow \text{worst-case time complexity}}$

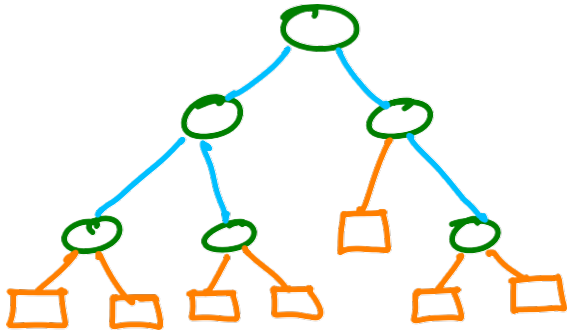
the SORTING LOWER BOUND

---

a lower bound for the worst-case time complexity of all comparison-based sorting algorithms

Consider the decision tree corresponding to any algorithm that can sort n items.

↳ e.g., the best algo EVER.



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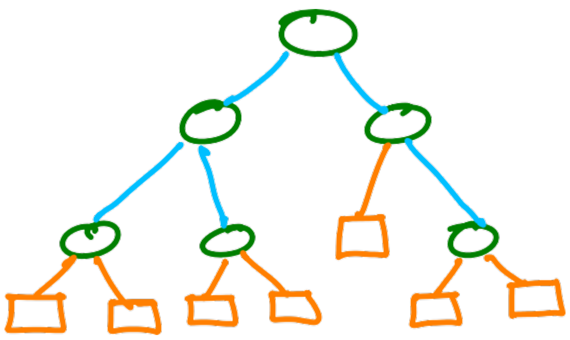

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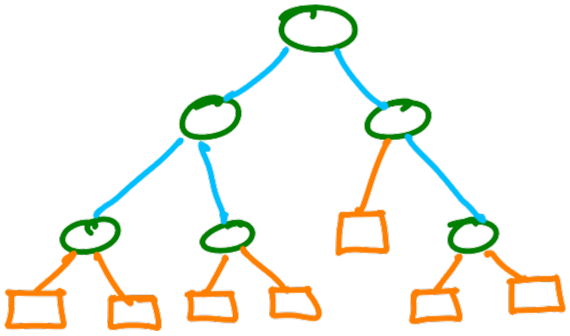

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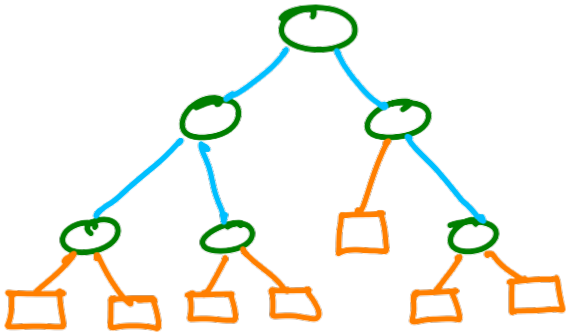

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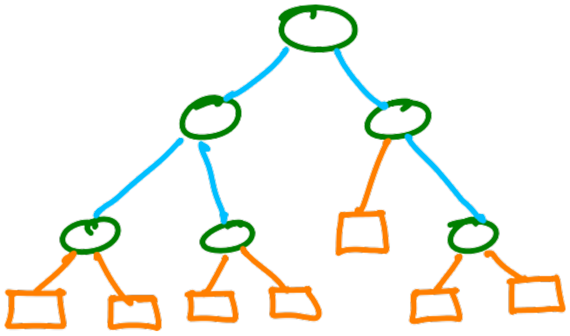

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To have $n!$ leaves, the tree needs a height of at least $\log_2(n!)$

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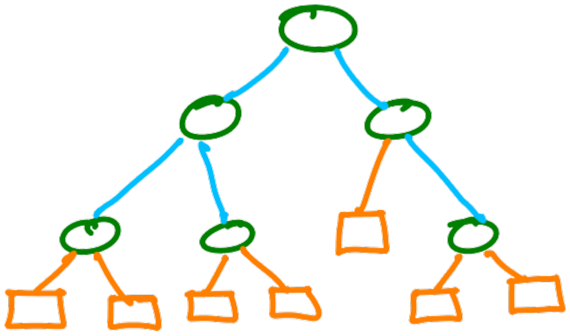

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Conclusion: For every comparison-sort algorithm,

worst-case time complexity $\geq \lceil \log_2 n! \rceil$

Approximating $\log_2 n!$

$\log(n!) = O(?)$

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$\log(n!) \leq \log(n^n)$

$$\log(n!) = O(?)$$

$$\log(n!) \leq \log(n^n) = n \log n$$

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$$\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdots 3 \cdot 2 \cdot 1)$$

$$\log(n!) = O(n \log n)$$

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$$\log(n!) = \Omega(?)$$

$$\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdots \cdots 3 \cdot 2 \cdot 1)$$

$$= \log(n \cdot 1 \cdot (n-1) \cdot 2 \cdot (n-2) \cdot 3 \cdot (n-3) \cdot 4 \cdots \cdots \sim (n-\tfrac{n}{2}) \cdot (n-\tfrac{n}{2}))$$

exactly if n: even

$$\log(n!) = O(n\log n)$$

$$\log(n!) \leq \log(n^n) = n\log n$$

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$$\log(n!) = \log(n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots\cdots 3\cdot 2\cdot 1)$$

$$= \log(\underbrace{n\cdot 1}\cdot\underbrace{(n-1)\cdot 2}\cdot\underbrace{(n-2)\cdot 3}\cdot\underbrace{(n-3)\cdot 4}\cdots\cdots\sim\underbrace{(n-\tfrac{n}{2})\cdot(n-\tfrac{n}{2})})$$

$$\geq \log(n\cdot n\cdot n\cdot n\cdot\ \cdots\ n)$$

(~ n/2 terms)

$$\log(n!) = O(n\log n)$$

$$\log(n!) \leq \log(n^n) = n\log n$$

$$\log(n!) = \Omega(?)$$

$$\log(n!) = \log\left(n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots\ \cdots 3\cdot 2\cdot 1\right)$$

$$= \log\left(\underbrace{n\cdot 1}\cdot\underbrace{(n-1)\cdot 2}\cdot\underbrace{(n-2)\cdot 3}\cdot\underbrace{(n-3)\cdot 4}\cdots\ \cdots \sim \underbrace{\left(n-\tfrac{n}{2}\right)\cdot\left(n-\tfrac{n}{2}\right)}\right)$$

$$\geq \log\left(n\ \cdot\ n\ \cdot\ n\ \cdot\ n\ \cdot\ \cdots\ n\right)$$

$$= \log\left(n^{n/2}\right)$$ (assume n: even otherwise $\lfloor \frac{n}{2} \rfloor$)

$$\log(n!) = O(n\log n)$$

$$\log(n!) \leq \log(n^n) = n\log n$$

$$\log(n!) = \Omega(n\log n)$$

$$\begin{aligned}
\log(n!) &= \log(n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots\cdots 3\cdot 2\cdot 1) \\
&= \log(\underbrace{n\cdot 1}\cdot\underbrace{(n-1)\cdot 2}\cdot\underbrace{(n-2)\cdot 3}\cdot\underbrace{(n-3)\cdot 4}\cdots\cdots\sim\underbrace{(n-\tfrac{n}{2})\cdot(n-\tfrac{n}{2})}) \\
&\geq \log(n\cdot n\cdot n\cdot n\cdot\cdots n) \\
&= \log(n^{n/2})
\end{aligned}$$

(assume n: even
otherwise $\lfloor\frac{n}{2}\rfloor$)

$$\Rightarrow \log(n!) \geq \frac{n}{2}\log n$$

$$\log(n!) = O(n \log n)$$

$$\log(n!) \leq \log(n^n) = n \log n$$

$$\log(n!) = \Omega(n \log n)$$

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$$\geq \log(n \cdot n \cdot n \cdot n \cdot \cdots n)$$

$$= \log(n^{n/2})$$ (assume n: even otherwise $\lfloor \frac{n}{2} \rfloor$) $\Rightarrow \log(n!) \geq \frac{n}{2} \log n$

---

so $\frac{1}{2} n \log n \leq \log(n!) \leq n \log n$