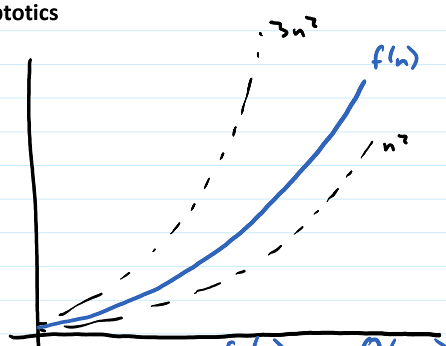
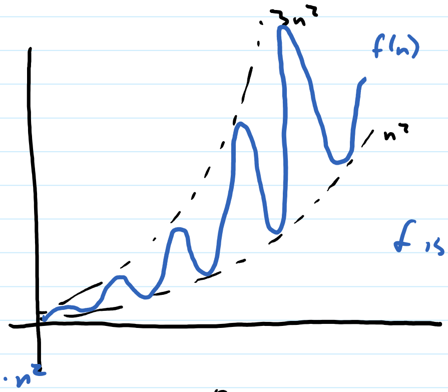
Asymptotics

$\lim_{n \to \infty} \frac{f(n)}{g(n)}$





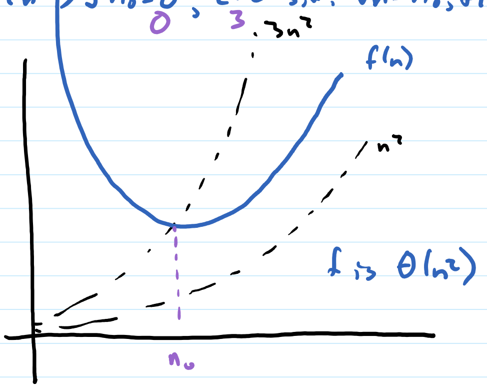
f is $\Theta(n^2)$

f(n) is $\Theta(n^2)$

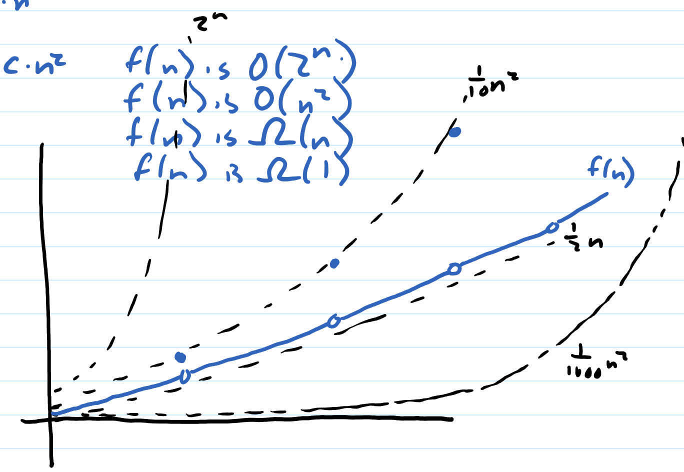
f is $\Omega(n^2)$ $\exists\, n_0 \geq 0,\ c > 0$ s.t. $\forall n \geq n_0,\ f(n) \geq c \cdot n^2$
$n_0 = 0$, $c = 1$

f is $O(n^2)$ $\exists\, n_0 \geq 0,\ c > 0$ s.t. $\forall n \geq n_0,\ f(n) \leq c \cdot n^2$
$n_0 = 0$, $c = 3$

f(n) is $O(2^n)$
f(n) is $O(n^2)$
f(n) is $\Omega(n)$
f(n) is $\Omega(1)$



f is $\Theta(n^2)$



THM: For any functions f, g, h, if f is $O(g)$ and g is $O(h)$, then f is $O(h)$

Proof: Suppose f, g, h are fxns s.t. f is $O(g)$ and g is $O(h)$ [want f is $O(h)$] $\rightarrow \exists\, n_0 \geq 0, c > 0$ s.t. $\forall n \geq n_0,\ f(n) \leq c \cdot h(n)$

Since f is $O(g)$, by def of O $\exists\, n_1 \geq 0$ and $c_1 > 0$ s.t. $\forall n \geq n_1,\ f(n) \leq c_1 \cdot g(n)$

and since g is $O(h)$ $\exists\, n_2 \geq 0$ and $c_2 > 0$ s.t. $\forall n \geq n_2,\ g(n) \leq c_2 \cdot h(n)$

Then $\forall n \geq \max(n_1, n_2),\ f(n) \leq c_1 \cdot c_2 \cdot h(n)$

So $\exists\, n_0 \geq 0,\ c > 0$ s.t. $\forall n \geq n_0,\ f(n) \leq c \cdot h(n)$
$n_0 = \max(n_1, n_2)$, $c = c_1 \cdot c_2$

So by def of O, f(n) is $O(h(n))$

Data Structures

input: M, W
PrefM[i][j] = machinist i's j^{th} fave welder
PrefW[i][j] = welder i's j^{th} fave machinist

initialize Next[i] = 1 for all i $\Theta(n)$
rank of next welder machinist i makes an invitation to

Initialize Rank[i][j] s.t.
Rank[i][j] = rank by welder i of machinist j $\Theta(n^2)$

init Match[i] = -1 (match[w] = m means (m,w) ∈ Tent)

FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}

Queue
check if empty $O(1)$

While there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m – look at head of queue $O(1)$
    let w be m's highest ranked s.t. (m,w) not in Invitations
        w = PrefM[m][Next[m]] $O(1)$
    add (m,w) to Invitations   Next[m]++

    Match[w] = -1
    if w in FreeW then
        remove w from FreeW
        remove m from FreeM – remove head $O(1)$
        add (m,w) to Tentative Match[w] = m $O(1)$
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM – remove head $O(1)$
            add m' to FreeM – add to end $O(1)$
            remove (m', w) from Tentative
            add(m, w) to Tentative Match[w] = m $O(1)$
return Tentative (built from Match) $\Theta(n)$

Rank[w] = [3, 1, 4, 2, 6, 5]
PrefW[w] = [2, 4, 1, 3, 6, 5]

does w prefer m to m'?

$\Theta(n^2)$ overall

Correctness

```
FreeM <- M
FreeW <- W
Invitations <- {}
Tentative <- {}

While there is an m in FreeM s.t. there is a w s.t. (m,w) not in Invitations
    choose such an m
    let w be m's highest ranked s.t. (m,w) not in Invitations

    add (m,w) to Invitations

    if w in FreeW then
        remove w from FreeW
        remove m from FreeM
        add (m,w) to Tentative
    else
        find m' s.t. (m', w) in Tentative
        if w prefers m to m'
            remove m from FreeM
            add m' to FreeM
            remove (m', w) from Tentative
            add(m, w) to Tentative
return Tentative
```

seq of matches for welder 3

$m_3 = NIL, NIL, NIL, 4, 4, \underbrace{4, 6}, 1, 1, \ldots$

Match[i] = m_{ij} after j iterations

Obs 1: Let m_{ij} be the machinist welder i is tentatively matched with after j iterations (or NIL if not matched yet)

a) Then if m invites w_i during iteration j, $m_{ij} \neq NIL$ and $m_{ik} \neq NIL$ for $k > j$

b) If $k > j$ and $m_{ij} \neq m_{ik}$ then welder i prefers m_{ik} to m_{ij}

Obs 2: $m \in FreeM \leftrightarrow$ there is no w s.t. $(m,w) \in Tentative$

$w \in Free\ W \leftrightarrow$ there is no m s.t. $(m,w) \in Tentative$

Obs 3: Tentative $\subseteq$ Invitations and is a matching

Obs 4: $m \in FreeM \rightarrow$ there is a w s.t. $(m, w) \notin Invitations$

Suppose $m \in FreeM$ but for all w $(m,w) \in Invites$

So by Obs 1, all w are matched

and by Obs 3, all n machinists are matched

So by Obs 2, no machinist is Free $\Rightarrow\Leftarrow$

When G-S terminates, Tentative is a perfect matching

2 ways alg can terminate: 1) There are free machinists who have made all invitations
2) No free machinists

1) can't happen b/c it contradicts Obs 4
2) No free machinists + Obs 2 & 3 = all machinists, welders matched so perfect matching

When G-S terminates, Tentative is a stable matching

Suppose not stable — there is an instability (m, w') wrt Tentative.

(m, w') $\notin$ Tentative by def. of instability

Tentative is a perfect matching, so can find w, m' s.t. (m, w), (m', w') $\in$ Tentative

m invited w (only way to get match (m, w)) m prefers w' to w (def of instability)

m invited w' (machinists go through welders in ↓ order of pref)

2 cases: i) w' rejected m immediately b/c matched with m'' who w' prefers to m. w' ended up matched with m', so either m' = m'' or w' prefers m' to m'' (obs 1b)

either way, w' prefers m' to m

ii) w' tentatively accepted m but later rejected in favor of some m'' that w' prefers to m; again ended up w/ m', so either m' = m'' or w' prefers m' to m'', so w' prefers m' to m

in both cases, w' prefers m' to m

but then (m, w') is not an instability ⇒⇐

∴ stable