# Shortest Paths

Dijkstra(G, ℓ)
  $S \leftarrow \{s\}$
  $d(s) \leftarrow 0$
  while $S \neq V$
    choose $v \notin S$ to minimize $d'(v) = \min_{\substack{u \in S \\ (u,v) \in E}} d(u) + \ell(u,v)$

    $S \leftarrow S \cup \{v\}$
    $d(v) = d'(v)$

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Q ← ∅
S ← {s}                         makes a) T
d[s] ← 0                        ] make d) T
π[s] ← NIL
for v ∈ V, v ≠ s
    if (s,v) ∈ E
        d'[v] ← ℓ(s,v)
        π[v] ← s                — makes e) T
    else                                g)
        d'[v] ← ∞
        π[v] ← NIL                      f)
    Q.enqueue(v, d'[v])         makes inv b),c) T
while Q ≠ ∅
    v = Q.extractMin()
    d[v] ← d'[v]
    S ← S ∪ {v}
    for (v,w) ∈ E where w ∈ Q
        if d[v] + ℓ(v,w) < d'[w]
            d'[w] = d[v] + ℓ(v,w)
            π[w] = v
            Q.decreasePriority(w, d'[w])
return (d, π)
```

INVARIANT:
a) $s \in S$
b) $S, Q$ partition $V$
c) $|Q| = n - 1 -$ # iterations
d) for $v \in S$, $d[v]$ = cost of min-cost path $s \rightsquigarrow v$
   $\pi[v]$ is next-to-last on that path
e) for $v \in Q$, $d'[v]$ = cost of min-cost path $s \rightsquigarrow v$ where intermediate verts are in $S$
   $\pi[v]$ = next-to-last vert on that path
f) $d'[v]$ is the priority of $v$ in $Q$
g) $\pi[v] = \text{NIL}$ or $\pi[v] \in S$ for all $v \in V$

Basis: ✓

Term: when loop stops, $Q = \emptyset$, so $S = V$ (inv b), so inv d) says $\forall v \in V$, $d, \pi$ are correct

Induction:
a) nothing removed from $S$
b) every time $v$ removed from $Q$, added $S$; those are only changes to $S, Q$
c) one extractMin per iteration

d) [Need for the one $v$ added to $S$, $d$ and $\pi$ are correct]
   ↳ cost of min-cost path $s \rightsquigarrow v$

   Consider any path $p$ $s \rightsquigarrow v$ [want cost of that path to be $\geq d[v]$]

   if $p$ has all int. verts in $S$ then
      $d'[v] \leq \ell(p)$ by INV e)

   else $p$ has int verts not in $S$
      find 1st edge $(x,y)$ that crosses from $S$ to $Q$

else $p$ has int verts not in $S$
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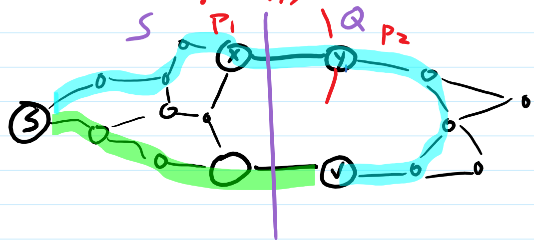



$d'[v] \le d'[y]$ (extract min + inv f)

split $p$ at $y$ into $p_1, p_2$

$d'[y] \le \ell(p_1)$ (inv e)
$\le \ell(p_1) + \ell(p_2)$ since $\ell(p_2) \ge 0$ (no neg weight edges)
$= \ell(p)$
$\ell(p_1) \le \ell(p)$

$d'[v] \le d'[y] \le \ell(p_1) \le \ell(p)$

so $d'[v]$ is $\le$ cost of any path $s \rightsquigarrow v$
and is correct value for $d[v]$ to be set to

f) any change to $d'$ is accompanied by corresponding change to $Q$

e) [$S$ expands, which expands the set of paths $d'$ needs to minimize over – $v$ is allowed as an intermediate vert — need to show that our updates to $d'$ are sufficient]

Consider any path $p$ $s \overset{S}{\rightsquigarrow} v \overset{S}{\rightsquigarrow} y \to w$
compare to $p'$ $\underbrace{s \overset{S}{\rightsquigarrow} y}_{\text{shortest}} \to w$ – verts all in $S$ by inv d, g

$\ell(p) \ge \ell(p') \ge d'_{old}[w] \ge d'_{new}[w]$

replaces $s \rightsquigarrow v \rightsquigarrow y$ with shortest $s \rightsquigarrow y$ inv e code

Consider any path $p$ $s \overset{S}{\rightsquigarrow} v \to w$
compare to $p'$ $\underbrace{s \overset{S}{\rightsquigarrow} v}_{\text{shortest}} \to w$

$\ell(p) \ge \ell(p') = d[v] + \ell(v,w) \ge d'_{new}(w]$

shortened code

Consider any path $p$ $s \overset{S}{\rightsquigarrow} w$

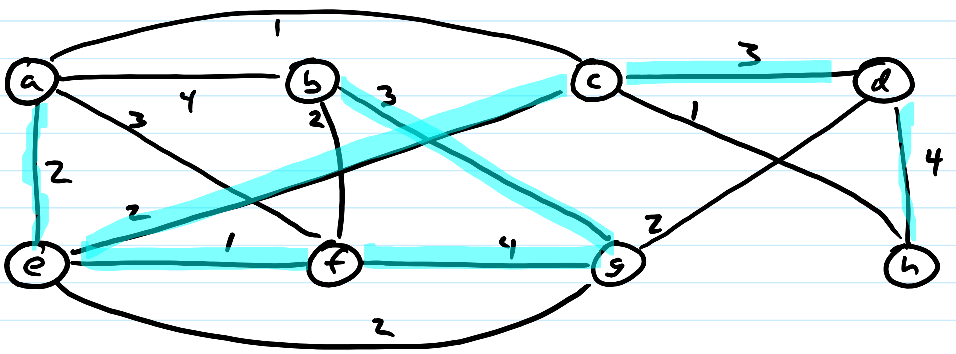
$\ell(p) \ge d'_{old}[w] \ge d'_{new}[inv]$

inv e code

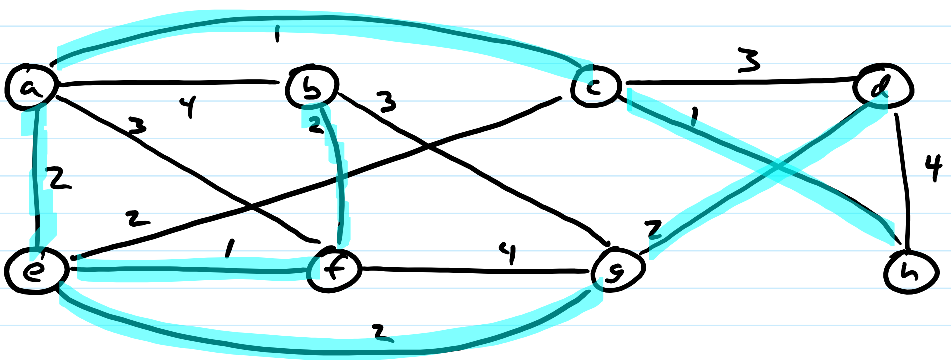
So $d'_{new}[w] \le \ell(p)$ for any path $s \rightsquigarrow w$ using int. verts in $S \cup \{v\}$

g) only time $\pi[w]$ is changed, it is changed to something just added to S;
nothing is ever removed from S

Minimum Spanning Tree : MST of undirected, weighted (nonnegative weights) graph G is a set of edges $T \subseteq E$ that forms a tree, connects all vertices in V, and minimizes total weight

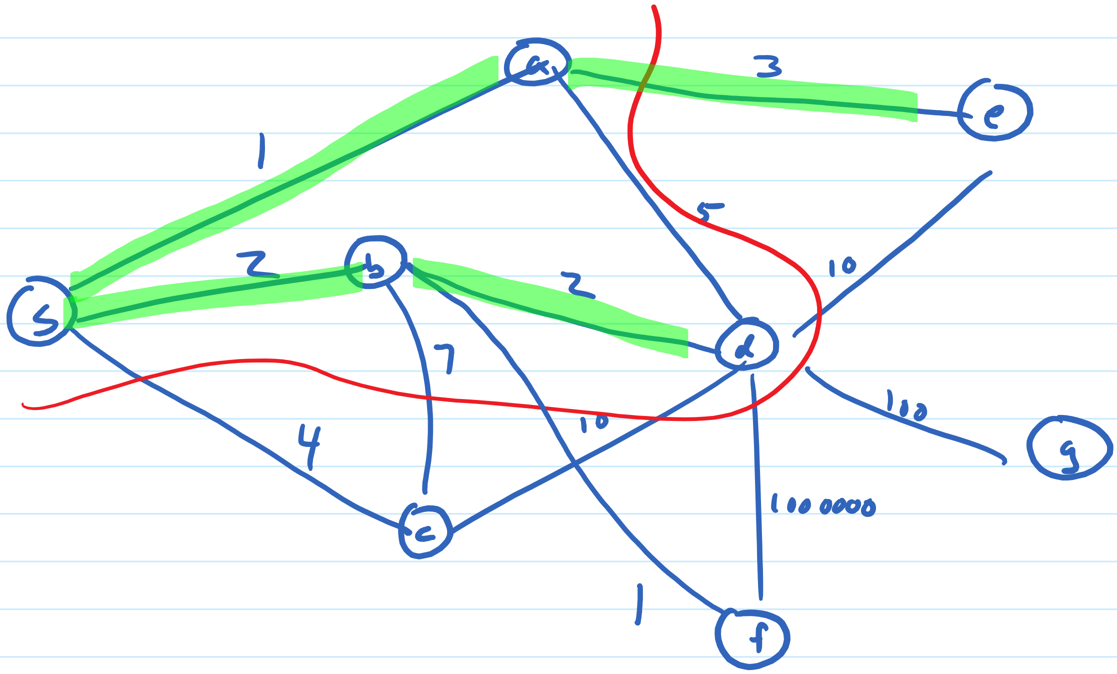



spanning tree
weight = 19




minimum spanning tree
weight = 11

**Prim's Algorithm**




Prim

$\cancel{\text{Dijkstra}}(G, \ell)$

$S \leftarrow \{s\}$

$d(s) \leftarrow 0$

while $S \neq V$

choose $v \notin S$ to minimize $d'(v) = \min\limits_{\substack{u \in S \\ (u,v) \in E}} \cancel{d(u) +}\, \ell(u,v)$

$Q \leftarrow \emptyset$

$S \leftarrow \{s\}$

$\cancel{d[s] \leftarrow 0}$

$\pi[s] \leftarrow \text{NIL}$

for $v \in V$, $v \neq s$

if $(s,v) \in E$

$d'[v] \leftarrow \ell(s,v)$

$\pi[v] \leftarrow v$

else

$d'[v] \leftarrow \infty$

$\pi[v] \leftarrow \text{NIL}$

$Q.\text{enqueue}(v, d'[v])$

while $Q \neq \emptyset$

$Q.\text{enqueue}(v, d'[v])$

while $Q \neq \emptyset$

$\quad v = Q.\text{extractMin}()$

$\quad \cancel{d[v] \leftarrow d'[v]}$ $\quad A \leftarrow A \cup \{(\pi[v], v)\}$

$\quad S \leftarrow S \cup \{v\}$

$\quad$ for $(v, w) \in E$ where $w \in Q$

$\quad\quad$ if $d\cancel{[v]} + \ell(v,w) < d'(w)$

$\quad\quad\quad d'[w] = d\cancel{[v]} + \ell(v,w)$

$\quad\quad\quad \pi[w] = v$

$\quad\quad\quad Q.\text{decreasePriority}(w, d'[w])$

return $(\cancel{d, \pi})$ $A$