# Counting Problems

k-Combination of $\{1, \dots, n\}$ is a size-$k$ subset of $\{1, \dots, n\}$

"n choose k" $\binom{n}{k}$ = # of k-combinations of $\{1, \dots, n\}$

$$= \begin{cases} 0 & \text{if } k > n \text{ or } k < 0 \\ 1 & \text{if } n = k \text{ or } k = 0 \\ \underbrace{\binom{n-1}{k-1}}_{\text{use } n} + \underbrace{\binom{n-1}{k}}_{\text{don't use } n} & \text{otherwise} \end{cases}$$

$\binom{4}{2} = 6$

$\{1,2,3,4\}$

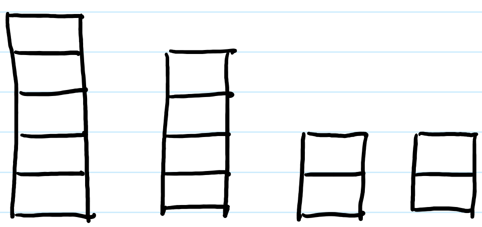
1,2    2,3
1,3    2,$\cancel{4}$
1,$\cancel{4}$    3,$\cancel{4}$

$\binom{3}{2}$ all the 2-combos of $\{1,2,3\}$

all the 1-combos of $\{1,2,3\}$

$\binom{3}{1}$ of these

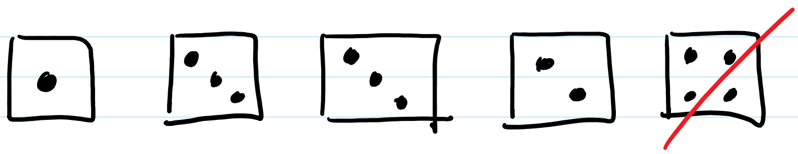
```
CHOOSE(n, k)
    if k<0 or k>n return 0
    C ← n×n array
    C[0,0] = 1
    for i=1 to n
        C[i,0] ← 1
        C[i,i] ← 1
        for j=1 to i-1
            C[i,j] ← C[i-1,j-1] + C[i-1,j]

    return C[n,k]
```



5    4    2    2

Count$(n,h)$ = # nonincreasing sequences of length $n$, height $h$

$$= \left\{ \right.$$



How many sequences of 5 die rolls sum to 13?

$$\underbrace{\text{Count}(n,k)}_{O(nk)\text{ entries}} = \#\text{ sequences of } n \text{ rolls that sum to } k$$

$$= \begin{cases} 0 & \text{if } k < n \text{ or } k > 6n \\ 1 & \text{if } k = 0 \text{ and } n = 0 \\ \underbrace{\sum_{i=1}^{6} \text{count}(n-1, k-1) \quad \text{otherwise}}_{O(1)\text{ time}} \end{cases}$$

$O(nk)$ total

# Seam Carving

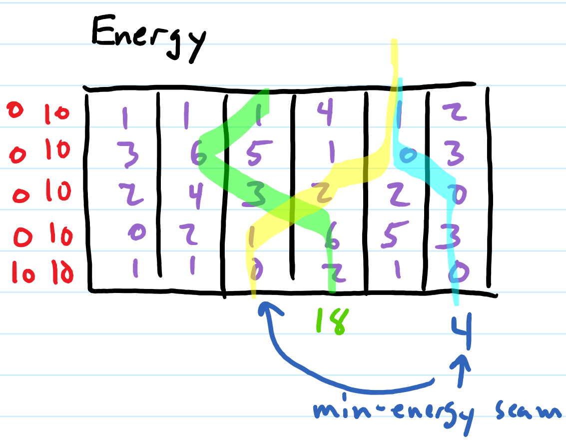
https://www.youtube.com/watch?v=6NclJXTlugc

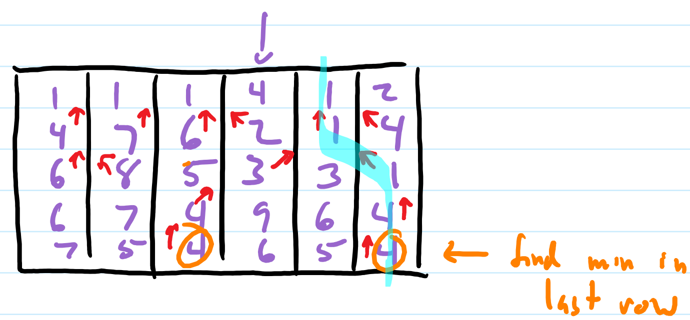
$M[i,j]$ = cost of min-cost seam from top to row $i$, col $j$

$O(n \cdot m)$ entries

$$= \begin{cases} E[i,j] & \text{if } i=0 \\ E[i,j] + \underbrace{\min(M[i-1,j-1], M[i-1,j], M[i-1,j+1])}_{O(1)} \end{cases}$$

$O(n \cdot m)$ total



## Subset Sum

Given set of positive integers $S$ and sum $k$, find $S' \subseteq S$ to maximize $\sum_{x \in S'} x$ s.t. $\sum_{x \in S'} x \le k$

Example: $S = \{1, 4, 9, 13, 16, 20\}$ $\quad k = 28$
$\quad x_1 \; x_2 \; x_3 \; \ldots$

Brute Force: try each $\underbrace{\text{subset}}_{2^n}$

Substructure: $\{1, 4, 9\}$ was subset closest to $28 - 13 = 15$ w/o going over among $\{1, 4, 9\}$

$\underline{OPT(i, w)}$ = highest possible sum $\le w$ using $1^{st}$ $i$ items

$O(n \cdot w)$ entries

$O(nw)$ total

$$= \begin{cases} 0 & \text{if } i = 0 \\ \cancel{\max_{j=1\ldots n} x_j + OPT(j-1, w-x_j)} & \\ \max\left(\underbrace{OPT(i-1, w-x_i) + x_i}_{\text{use element } i},\ OPT(i-1, w)\right) & \text{if } x_i \le w \\ OPT(i-1, w) & \text{otherwise} \end{cases}$$

$O(1)$ per entry

| | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 2 | 0 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | - | - | - | - | | |
| 9 | 3 | 0 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 5 | 9 | 10 | 10 | 10 | 13 | 14 | 14 | 14 | 14 | - | - | - | | | | | | | | |
| 13 | 4 | 0 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 5 | 9 | 10 | 10 | 10 | 13 | 14 | 14 | 14 | 17 | 18 | 18 | 18 | 18 | 22 | 23 | 23 | 23 | 26 | 27 | 27 |
| 16 | 5 | 0 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 5 | 5 | 10 | 10 | 10 | 13 | 14 | 14 | 14 | 17 | 18 | 18 | 20 | 21 | 21 | 21 | 21 | 25 | 26 | 27 | 27 |
| 20 | 6 | 0 | 1 | 1 | 1 | 4 | 5 | 5 | 5 | 5 | 5 | 10 | 10 | 10 | 13 | 14 | 14 | 14 | 17 | 18 | 18 | 20 | 21 | 21 | 21 | 24 | 25 | 26 | 27 | 27 |

$S = \{1, 4, 9, 13, 16, 20\}$ $k = 23$

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | | | | | | | | | | |
| 9 | | | | | | | | | | | | | | | | | | | | | | | | |
| 13 | | | | | | | | | | | | | | | | | | | | | | | | |
| 16 | | | | | | | | | | | | | | | | | | | | | | | | |
| 20 | | | | | | | | | | | | | | | | | | | | | | | | |

## Vendor Scheduling

| | Week 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| Brooklyn | 6 | 3 | 10 | 4 | 1 | M = 10 |
| Stratford | 4 | 6 | 3 | 8 | 20 | |

https://play.golang.org/p/NgB2MxoSmX

opt $\underbrace{S \quad S \quad S \quad S}_{\text{OPT for } 1^{st} \text{ 4 weeks?}}$, S  41

NO :( BBBB = 23 > 21

$OPT_B(n)$ = best net through week n ending in B

$OPT_S(n)$ = " " ending in S

$$OPT_B(n) = \begin{cases} B_1 & \text{if } n=1 \\ \max\left(OPT_S(n-1) - m,\ OPT_B(n-1)\right) + B_n & \end{cases}$$