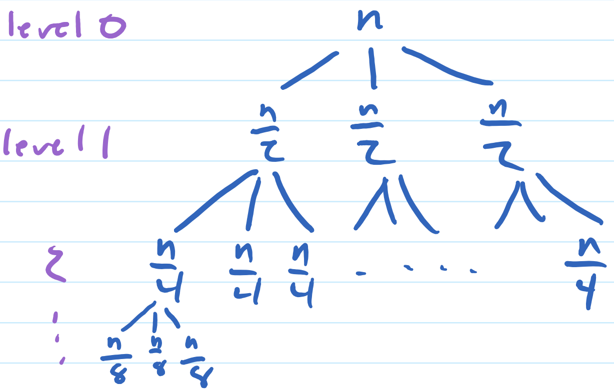
Recursion tree for $T(n) \le 3T(\frac{n}{2}) + cn$



@ level i
work in
divide+combine
$= 3^i \cdot c\frac{n}{2^i}$

cn

$3 \cdot \frac{cn}{2}$

$9 \cdot \frac{cn}{4}$

$\vdots$

$$\sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i \cdot cn$$

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1}$$

$$= cn \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i$$

$$= 2cn \left(\frac{3}{2}^{\log_2 n + 1} - 1\right)$$

$$\le 2cn \, \frac{3}{2}^{\frac{\log_{\frac{3}{2}} n}{\log_{\frac{3}{2}} 2} + 1}$$

$$x^{a \cdot b} = (x^a)^b$$

$$\log_b a = \frac{1}{\log_a b}$$

$$= 2cn \, \frac{3}{2}^1 \cdot \left(\frac{3}{2}^{\log_{\frac{3}{2}} n}\right)^{\frac{1}{\log_{\frac{3}{2}} 2}}$$

$$= 3\,cn \cdot n^{\log_2 \frac{3}{2}}$$

$$= 3cn \cdot n^{(\log_2 3 - \log_2 2)}$$

$$= 3\,cn \cdot n^{\log_2 3 - 1}$$

$$= 3c\cancel{n} \cdot n^{\log_2 3} \cdot \cancel{\frac{1}{n}}$$

$$= 3c\,n^{\log_2 3}$$

$\log_2 3 < 2$

which is $O(n^{\underline{\log_2 3}})$

Recursion Tree for $T(n) \le \overset{a}{\underset{\cancel{10}\ 9}{\cancel{2}}}\, T\left(\frac{n}{\underset{b}{3}}\right) + cn^{\overset{d}{2}}$

cn^2

$\frac{10}{9} cn^2$

$\frac{100}{81} cn^2$

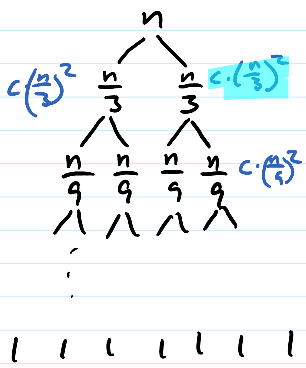
$\left(\frac{10}{9}\right)^i cn^2$

$cn^2 \cdot \sum_{i=0}^{\log_3 n} \left(\frac{10}{9}\right)^i$

$\le$ ⋮

$O\left(n^{\log_3 10}\right)$

$d = 2$



cn^2

$\frac{2}{9} \cdot cn^2$

$\frac{4}{81} cn^2$

⋮

at level i: $\left(\frac{2}{9}\right)^i cn^2$

$$\text{sum} = \sum_{i=0}^{\log_3 n} \left(\frac{2}{9}\right)^i \cdot cn^2$$

$$\le \sum_{i=0}^{\infty} \left(\frac{2}{9}\right)^i \cdot cn^2$$

$$= cn^2 \cdot \sum_{i=0}^{\infty} \left(\frac{2}{9}\right)^i$$

$$= \frac{9}{7} cn^2$$

$$O(n^2)$$

$$\sum_{i=0}^{\log_3 n} \left(\frac{9}{9}\right)^i \cdot cn^2$$

$$= O(n^2 \log n)$$

$$\sum \left(\frac{a}{b^d}\right)^i c \cdot n^d$$

which of 3 cases depends on

$$\frac{a}{b^d} < \overset{?}{=} > 1$$

$$\log_b a < \overset{?}{=} > d$$

Master Method

Suppose $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ $\lfloor\,\rfloor, \lceil\,\rceil, \pm 1, \ldots$ don't matter

Then

if $f(n) \in O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$ then $T(n)$ is $O(n^{\log_b a})$

if $f(n) \in \Theta(n^{\log_b a})$ then $T(n)$ is $O(f(n) \cdot \lg n)$

if $f(n) \in \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$ and if $a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ for some $c > 1$ and all large n

$T(n)$ is $O(f(n))$

Examples: $T(n) = 9T\left(\frac{n}{3}\right) + n$ $a = 9$ $b = 3$ $\log_b a = 2$

n is $O(n^{2 \overset{\varepsilon}{-0.1}})$

1st case $T(n)$ is $O(n^2)$

$T(n) = T\left(\frac{2}{3}n\right) + 1$ $a = 1$ $b = \frac{3}{2}$ $\log_{\frac{3}{2}} 1 = 0$ 1 is $\Theta(n^{\log_b a})$

$T(n)$ is $O(\log n)$

$T(n) = 3 \cdot T\left(\frac{n}{4}\right) + n \log n$ $a = 3$ $b = 4$ $\log_b a = \log_4 3$

$n \log n$ is $\Omega(n^{\log_4 3 + 0.1})$

3rd case $T(n)$ is $O(n \log n)$

$T(n) = 2T\left(\frac{n}{2}\right) + n \log n$ $a = 2$ $b = 2$ $\log_b a = 1$

$n \log n \in \Theta(n)$? NO

$n \log n \in O(n^{1-\varepsilon})$? NO

$n \log n \in \Omega(n^{1+\varepsilon})$? NO

is $\cancel{n} \log n$ $\Omega(n^{1.001})$

$n \log n \in \Omega(n \quad)$? NO

is $\cancel{n} \log n \quad \Omega(n^{1.001})$

$$\cancel{n} \cdot \sqrt[1000]{n}$$

MM does not apply
(this version)

Matrix Multiplication

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 4 & 6 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \; \end{pmatrix}$$

$O(n^3)$ from 202 method

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ \dots & \dots \end{pmatrix}$$

$S_1 = B_{12} - B_{22}$

$S_2 = A_{11} + A_{12}$

$S_3 = A_{21} + A_{22}$

$S_4 = B_{21} - B_{11}$

$S_5 = A_{11} + A_{22}$

$S_6 = B_{11} + B_{22}$

$S_7 = A_{12} - A_{22}$

$S_8 = B_{21} + B_{22}$

$S_9 = A_{11} - A_{21}$

$S_{10} = B_{11} + B_{12}$

$P_1 = A_{11} \cdot S_1$

$P_2 = S_2 \cdot B_{22}$

$P_3 = A_{22} \cdot S_4$

$P_4 = S_3 \cdot B_{12}$

$P_5 = S_5 \cdot S_6$

$P_6 = S_7 \cdot S_8$

$P_7 = S_9 \cdot S_{10}$

$T(n) = 7 \cdot T(\frac{n}{2}) + cn^2$

n^2 is $O(n^{\log_2 7 - \varepsilon})$

$T(n)$ is $O(n^{\log_2 7})$

$C_{11} = P_5 - P_3 - P_2 + P_6$

$C_{12} = P_1 + P_2 = A_{11}(B_{12} - B_{22}) + (A_{11} + A_{12})B_{22}$

$= A_{11} B_{12} - \cancel{A_{11} B_{22}} + \cancel{A_{11} B_{22}} + A_{12} \cdot B_{22}$

$C_{21} = P_3 + P_4$

$C_{22} = P_5 + P_1 - P_3 - P_7$

result $= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

$$\text{result} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

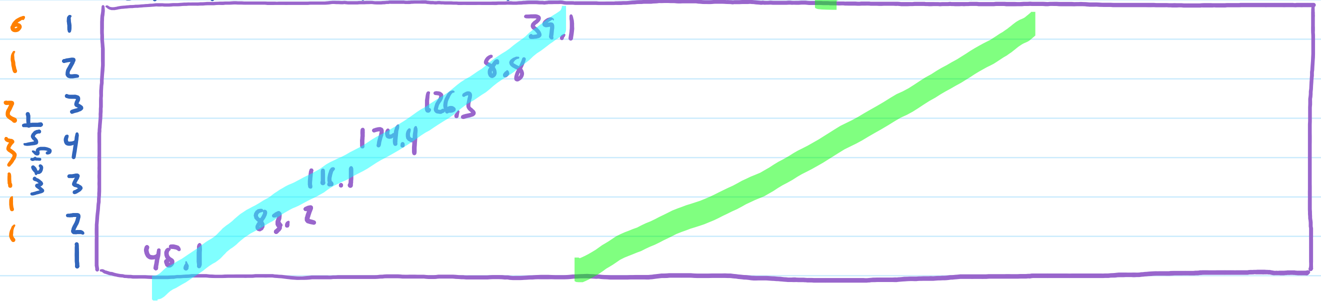
Convolution

Smoothing

avg high by date

Day	362 ... 364	0	1	...	59	60	61	62	63	64	65	...	364	0	1	2
avg					51.1	53.4	52.9	54.1	52.1	49.9	52.5					
weight			1		2	3	4	3	2	1						

	0	1	2	3																		
day	362	363	364	0	1	2	3	....	59	60	61	62	63	64	65	...	364	0	1	2
observation	45.1	41.6	38.7	43.6	42.1	44.4	39.1		51.1	53.4	52.9	54.1	52.1	49.9	52.5					



Convolution: $(a_0 \ a_1 \ \cdots \ a_{370}) * (b_0 \ b_1 \ \cdots \ b_6)$

$$= (a_0 b_0 \quad a_0 b_1 + a_1 b_0 \quad a_0 b_2 + a_1 b_1 + a_2 b_0 \ \cdots \ a_0 \cdot b_6 + \cdots + a_6 b_0 \quad a_1 b_6 + \cdots + a_6 b_1 \ \cdots \ a_{n-1} b_{m-1})$$

Multiplying Polynomials

coefficients

$A(x) = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$ $\quad A = (a_n \quad a_{n-1} \quad \cdots \quad a_0)$

$B(x) = (b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x_1 + b_0)$ $\quad B = (b_n \quad b_{n-1} \quad \cdots \quad b_0)$

$$C(x) = A(x) \cdot B(x)$$
$$= a_n b_n \cdot x^{2n} + (a_{n-1} b_n + a_n b_{n-1}) x^{2n-1} + \cdots + (a_0 b_1 + a_1 b_0) x + a_0 b_0$$

$$(a_n b_n \quad a_{n-1} b_n + a_n b_{n-1} \cdots a_1 b_0 + a_0 b_1 + a_0 b_0)$$
$$\parallel$$
$$A * B$$

Define $p(A)$ = polynomial with coefficients from A $\quad p(a_0, \ldots, a_{n-1}) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$

$$\underset{\substack{\uparrow \\ \text{poly mult}}}{p(A) \cdot p(B)} = p(\underset{\substack{\uparrow \\ \text{convolution}}}{A * B})$$

p is 1-1 and onto (bijective) $\quad p^{-1}(f)$ = vector of coefficients of f

```
CONVOLUTION(A, B)
    C ← p(A)·p(B) = p(A*B)
    return p^{-1}(C)        p^{-1}(C) = p^{-1}(p(A*B)) = A*B
```

can be done using any representation of the polynomials!
coefficients or points - for poly of degree k, any set of ≥ k+1 pts is sufficient

(x_1, y_1) is on P_1
(x_1, y_2) is on P_2
$(x_1, y_1 \cdot y_2)$ is on $P_1 \cdot P_2$

$A = x^2 - 5x + 6$
$B = x^2 - 1$
$A \cdot B = x^4 - 5x^3 + 5x^2 + 5x - 6$

(0, 6)	(1, 2)	(2, 0)	(3, 0)	(4, 2)
(0, -1)	(1, 0)	(2, 3)	(3, 8)	(4, 15)
(0, -6)	(1, 0)	(2, 0)	(3, 0)	(4, 30)

vector A → coefficients of p(A) → 2n pts on p(A)

O(n^2) by normal eval
O(n log n) by divide/conquer

vector B → coefficients of p(B) → 2n pts on p(B)

O(n)

→ 2n pts on p(A)·p(B) → coefficients of p(A)·p(B)

O(n log n) by divide/conquer

$\parallel$
p(A * B)

O(1) ↓

vector A * B

```
CONVOLUTION(A, B)
    return POLY-MULT(A, B)
```

n = len(A)
↙ A is degree n-1 $\quad$ A·B is degree 2n-2; need ≥ 2n-1 pts; choose 2n (WHY?)

```
POLY-MULT(A, B)
    Ax = POLY_EVAL(A, 2n)        O(n log n)
    Bx = POLY_EVAL(B, 2n)        O(n log n)
    Cx = Ax · Bx                 O(n)
    return INTERPOLATE(Cx)       O(n log n)
```

$2\pi i$

```
Cx = Ax · Bx                          O(n)
return INTERPOLATE(Cx)                O(n log n)
```

```
POLY_EVAL(A, n)
    if degree(A) = 0
        return [a0, a0, ..., a0]
    else
        EVEN ← [a0, a2, a4, ...]
        ODD ← [a1, a3, ...]
        Ex ← POLY_EVAL(EVEN, n)
        Ox ← POLY_EVAL(ODD, n)
        for j = 0 to k-1
            RESULT[j] = Ex[2j mod k] + $e^{-2\pi i\frac{k}{n}}$ · Ox[2j mod k]
        return RESULT
```

returns A evaluated at kth roots of 1 $A(1), A(z), A(z^2), \ldots, A(z^{k-1})$

$$\overset{e^{\frac{2\pi i}{k}}}{\downarrow}\ z$$

or [0] if degree(A)=1

$$A(x)=E(x^2)\cdot \underset{e^{2\pi i\frac{j}{k}}}{\underset{\downarrow}{x}}\cdot \overbrace{O(\underbrace{x^2}_{e^{2\pi i\frac{2j}{k}}})}^{O_x[2j \bmod k]}$$

$$T(n) \leq 2\cdot T(\tfrac{n}{2}) + c\cdot n$$

$T(n)$ is $O(n \log n)$